TOWARDS A THETA CORRESPONDENCE IN FAMILIES FOR TYPE II DUAL PAIRS

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1. Introduction

In the theory of automorphic forms, the theta correspondence occupies a significant role as it allows to build concrete automorphic representations. Over a local non-archimedean field of characteristic different from 2, the local theta correspondence involves two groups that form a dual pair in a symplectic group and provides a bijection between prescribed subsets of irreducible representations of (central extensions of) these two groups. This bijection encodes some deep arithmetic information, with an important range of applications. In the local setting, most of these applications lie into the framework of complex smooth representations of (connected) reductive groups over a local non-archimedean field.

It is an interesting question to extend results for more general coefficient fields. The study of representation of p-adic over fields of characteristic $\ell \neq p$ was initiated by Marie-France Vignéras and she started studying them because of the relations they have with congruences between automorphic forms. In this context, we talk about ℓ-modular representation. The same questions but involving coefficient fields of characteristic p requires complete different techniques and is not considered in this work, so when we use the word modular we will always mean ℓ-modular. In the recent years, there has been a growing interest in considering more general coefficients such as commutative $\mathbb{Z}[1/p]$-algebras and we refer to this situation as in families. These representations in families have been thoroughly studied by David Helm over $W(\overline{\mathbb{F}_\ell})$ for general linear groups and together with Emerton they formulated a local Langlands correspondence in families for general linear groups which was supplemented by some the work of Helm and the first author.

The theta correspondence deals with dual pairs that divides into two main kinds: type I and type II. The first one involves isometry groups such as symplectic, orthogonal and unitary, and the second involves general linear groups. The main statement of the theta correspondence yields a bijection between subsets of irreducible complex representations of two groups forming the dual pairs and this bijection is defined by the interplay of the Weil representation. For type II dual pairs, that is for two general groups over division algebras of finite dimension over a non-archimedean local field F, this bijection is due to some unpublished work of Howe and the proof can be found in the appendix of [Mín06]. One can ask about the validity of this bijection for modular representations, which is the starting point for the thesis work of Mínguez [Mín06]. He proved that as long as the characteristic ℓ of the coefficient field is not dividing the pro-orders of the groups forming a given type II dual pairs, this bijection still holds. See our Section 4.1 for the precise formulation of this bijection. However, there is no bijection when the characteristic divides the pro-orders of the group forming the dual pairs as a counter-example found by Mínguez shows.

1.1. Theta correspondence for type II dual pairs.

Let F be a non-archimedean local field of residual characateristic p and residual cardinal q. Note that F can be of characteristic 2 in this setting, as opposed to the type I case. Let n and m be two positive integers with $n \geqslant m$ and set $G_n = \mathrm{GL}_n(F)$ as well as $G_m = \mathrm{GL}_m(F)$.

When R is an algebraically closed field whose characteristic ℓ is not dividing the pro-orders of G_n and G_m, we say that ℓ is banal with respect to G_n and G_m. In this banal situation, the

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result of Mínguez states as an injective map:

$$\theta_R : \mathrm{Irr}_R(G_m) \to \mathrm{Irr}_R(G_n)$$

defined by the so-called Weil representation. This map is explicit in terms of Langlands quotient's classification of irreducible representations.

The supercuspidal support yields a meaningful partition of $\mathrm{Irr}_R(G_n)$ which we sum up as a surjective map $\mathrm{scs} : \mathrm{Irr}_R(G_n) \to \Omega_R(G_n)$. Of course a similar map for G_m exists. It turns out that the map θ_R is compatible with supercuspidal supports. More prosaically it means there exists an injective map $\Omega_R(G_m) \to \Omega_R(G_n)$ – which we still call θ_R – such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Irr}_R(G_m) & \xrightarrow{\theta_R} & \mathrm{Irr}_R(G_n) \\ \downarrow{\scriptstyle \mathrm{scs}} & & \downarrow{\scriptstyle \mathrm{scs}} \\ \Omega_R(G_m) & \xrightarrow{\theta_R} & \Omega_R(G_n). \end{array}$$

In addition this map is compatible with the Bernstein decompositions, which can be viewed as two partitions $\mathcal{B}_R(G_m)$ and $\mathcal{B}_R(G_n)$ that are coarser than the supercuspidal supports. This is alternatively known as the inertial (supercuspidal) support.

When the characteristic of ℓ divides the pro-orders of G_n and G_m, there is no way to define an injective map between irreducible representations as above. However, an injective map between supercupsidal supports can be implictly derived from the work of Mínguez [Mín08]. We explain now the kind of algebraic structure we are considering to interpret these maps.

1.2. **Geometrisation of θ_R.** The set of supercuspidal supports $\Omega_{\mathbb{C}}(G_m)$ enjoys a richer structure of (a disjoint union of) algebraic varieties following the results of Bersntein. To be more precise, the center $Z_{\mathbb{C}}(G_m)$ of the category $\mathrm{Mod}_{\mathbb{C}}(G_m)$ is acting on $\pi \in \mathrm{Irr}_{\mathbb{C}}(G_m)$ by a character $\eta_\pi : Z_{\mathbb{C}}(G_m) \to \mathbb{C}$ as a result of Schur's lemma. Then $\pi \in \mathrm{Irr}_{\mathbb{C}}(G_m) \mapsto \eta_\pi \in \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(Z_{\mathbb{C}}(G_m), \mathbb{C})$ turns out to be a surjective map whose underlying partition agree with the supercuspidal supports. Via this identification, supercuspidal supports correspond to $\mathbb{C}$-points of $X_{n,\mathbb{C}} = \mathrm{Spec}(Z_{\mathbb{C}}(G_m))$. There exists a decomposition of the center into a product, indexed by the Bernstein components, of integral domains that are finite type $\mathbb{C}$-algebras:

$$Z_{\mathbb{C}}(G_m) = \prod_{\mathfrak{s} \in \mathcal{B}_{\mathbb{C}}(G_m)} Z_{\mathbb{C}}^{\mathfrak{s}}(G_m).$$

As a consequence of this decomposition $X_{n,\mathbb{C}}$ is the disjoint union of the algebraic varieties $X_{n,\mathbb{C}}(\mathfrak{s}) = \mathrm{Spec}(Z_{\mathbb{C}}^{\mathfrak{s}}(G_m))$. The set of Bernstein components $\mathfrak{s}$ corresponds bijectively to the set of primitive idempotents e in $Z_{\mathbb{C}}(G_m)$ via the relation $Z_{\mathbb{C}}^{\mathfrak{s}}(G_m) = eZ_{\mathbb{C}}(G_m)$.

Thanks to the work of Vignéras, all these properties except one carry to algebraically closed fields R of characteristic $\ell \neq p$. The point that fails is that the rings $Z_R^{\mathfrak{s}}(G_m)$ are not necessarily reduced when the characteristic ℓ of R divides the pro-order of G_m. This latter fact constitutes a great obstacle when looking at good algebraic candidates $X_{m,R} \to X_{n,R}$ to interpolate θ_R. Indeed, the rings $Z_R^{\mathfrak{s}}(G_m)$ are Jacobson, so they are reduced if and only if the map:

$$i_R^{\#}(\mathfrak{s}) : Z_R^{\mathfrak{s}}(G_m) \to \prod_{\mathfrak{m}} Z_R^{\mathfrak{s}}(G_m)/\mathfrak{m}$$

where the product runs over all maximal ideals, is injective. As $Z_R^{\mathfrak{s}}(G_m)/\mathfrak{m} = R$ by Hilbert's nullstellensatz, any R-algebra morphism $Z_R(G_n) \to Z_R(G_m)$ is fully determined by its compisition with $i_R^{\#} = \prod i_R^{\#}(\mathfrak{s})$ as long as the ring $Z_R(G_m)$ is reduced. In terms of schemes, this reducedness condition guarantees that there exists at most one $X_{m,R} \to X_{n,R}$ that can agree with θ_R on R-points. Of course this uniqueness fails if we remove the reducedness assumption. Furthermore, when there are several candidates $X_{m,R} \to X_{n,R}$, it is not clear which one is best.

We change strategy to get these maps altogether as coming from a single one defined integrally *i.e.* over $\mathbb{Z}[1/p]$. The right map to consider over R will be the scalar extension of this one map, as well as the unique good candidate in the most favourable situation. Unfortunately there is no formal reason for such an integral map to exist, so we have got to consider a more thorough description of the Weil representation with coefficients in $\mathbb{Z}[1/p]$.

1.3. **Action of the Bernstein center on the Weil representation.** Let A be a commutative $\mathbb{Z}[1/p]$-algebra, so that p is invertible in A. The group $G_n \times G_m$ acts by matrix multiplication on the set of matrices $\mathcal{M}_{n,m}(F)$ via $(g_n, g_m) \cdot x = g_n x g_m^{-1}$. The Weil representation with coefficients in R, denoted by $\omega_{n,m}^A$, is the module of A-valued locally constant and compactly supported functions $C_c^\infty(\mathcal{M}_{n,m}(F), A)$ endowed with the linear action of $G_n \times G_m$ given by matrix multiplication:

$$(\omega_{n,m}^A(g_n, g_m) \cdot f)(x) = f(g_n^{-1} x g_m).$$

Assume that $n \geqslant m$. The rank k matrices $\mathcal{O}_{n,m}^k$ induces a stratification of $\mathcal{M}_{n,m}(F)$, as well as a filtration:

$$0 \subseteq \omega_{n,m}^{(m)} \subseteq \cdots \subseteq \omega_{n,m}^{(1)} \subseteq \omega_{n,m}^{(0)} = \omega_{n,m}^A$$

whose subquotients in the descending order are $W_{n,m}^k = C_c^\infty(\mathcal{O}_{n,m}^k, A)$ for $0 \leqslant k \leqslant m$. One of our main observations, based on the geometric lemma, is the following:

Proposition 1.1. *There is a canonical isomorphism:*

$$\mathrm{End}_{A[G_n \times G_m]}(W_{n,m}^k) \simeq Z_A(G_k).$$

Furthermore $\mathrm{Hom}_{A[G_n \times G_m]}(W_{n,m}^k, W_{n,m}^{k'}) = 0$ *if* $k < k'$.

As a result of this proposition, we can consider the natural ring morphisms:

$$\varphi_{n,m}^k : Z_A(G_n) \otimes_A Z_A(G_m) \to \mathrm{End}_{A[G_n \times G_m]}(W_{n,m}^k)$$

whose image lie naturally in $Z_A(G_k)$. Also define:

$$\varphi_{n,m} : Z_A(G_n) \otimes_A Z_A(G_m) \to \mathrm{End}_{A[G_n \times G_m]}(\omega_{n,m}^A).$$

Actually, there is a nice comparison result for the kernels of these maps as they all factor through $\varphi_{n,m}^m$ in the following sense:

Proposition 1.2. *We have a sequence of inclusions* $\ker(\varphi_{n,m}^m) \subseteq \cdots \subseteq \ker(\varphi_{n,m}^1) \subseteq \ker(\varphi_{n,m}^0)$. *In particular* $\ker(\varphi_{n,m}) = \ker(\varphi_{n,m}^m)$. *Furthermore there exists a map:*

$$\theta_{n,m}^\# : Z_A(G_n) \to Z_A(G_m)$$

induced by $\varphi_{n,m}^m$ *such that this common kernel is the ideal:*

$$\langle z \otimes_A 1 - 1 \otimes_A \theta_{n,m}^\#(z) \ | z \in Z_A(G_m) \rangle.$$

When $n = m$ *the map* $\theta_{n,n}$ *is the contragredient involution* $z \mapsto z^\vee$ *of the center* $Z_A(G_n)$.

This proposition claims that the center $Z_A(G_n)$ is acting through $\theta_{n,m}^\#$ on the Weil representation. Denoting $\theta_{n,m} : X_{m,A} \to X_{n,A}$ the associated map of affine schemes, we can consider for all commutative A-algebra B the induced map on B-points *i.e.* $\theta_{n,m}(B) : X_{m,A}(B) \to X_{n,A}(B)$ which is defined by composition $\mathrm{Spec}(B) \to X_{m,A} \to X_{n,A}$.

Because the Weil representation is compatible in an obvious way with scalar extension, we obtain that our map $\theta_{n,m}^\#$ is compatible to scalar extension and therefore our construction automatically gives the right $\theta_{n,m}(R)$ when R is an algebraically closed field of characteristic ℓ not dividing the pro-order of G_m. When ℓ is dividing the pro-order of G_m, it also provides a map $\theta_{n,m}(R)$ corresponding to $\theta_{n,m}^\#(R)$. Our choice of a good candidate in this situation is obtained thanks to the map $\theta_{n,m}^\#$ which has a natural interpretation in terms of (the endormorphisms of) the Weil representation. Note that our results can be compared to the results of Mínguez, but are independent of them and actually do not assume the theta correspondence.

1.4. **New features in our setting.** We can now study these $\theta^{\#}_{n,m}$ when R is an algebraically closed field. From the description of the explicit map on R-points, we obtain:

Proposition 1.3. *Assume that the characteristic ℓ of R is banal with respect to G_n and G_m. Then $\theta^{\#}_{n,m} : Z_R(G_n) \to Z_R(G_m)$ is surjective i.e. $\theta_{n,m} : X_{m,R} \to X_{n,R}$ is a closed immersion.*

Moreover, the maps $\theta^{\#}_{n,m}$ actually satisfy inductive relations:

Proposition 1.4. *For all $m \leqslant k \leqslant n$, we have:*

$$\theta^{\#}_{n,m} = \theta^{\#}_{k,m} \circ \theta^{\#}_{k,k} \circ \theta^{\#}_{n,k}.$$

2. Proof of Proposition 1.1

Our main tool to prove Proposition 1.1 is the so-called geometric lemma. In order to give more details about this technique, as well as not breaking the pace of reading for someone already familiar with that, we give a separate and detailed account in Appendix A. We simply recall some notations. Choose as a minimal parabolic subgroup of G_n, also called a Borel subgroup in this situation, the subgroup of upper triangular matrices B_n with Levi decomposition $T_n N_n$ where T_n is the subgroup of diagonal matices in G_n and N_n the set of unipotent matrices in B_k. For $0 \leqslant k \leqslant n$, set:

$$M_k^n = \left\{ \begin{bmatrix} a_k & 0 \\ 0 & b_{n-k} \end{bmatrix} \in G_n \;\middle|\; a_k \in G_k \text{ and } b_{n-k} \in G_{n-k} \right\}.$$

It is a standard Levi of G_n, which is contained in a unique standard parabolic subgroup denoted by $P_k^n = M_k^n N_k^n$. Let $Q_k^n = M_k^n \bar{N}_k^n$ be the opposite parabolic to P_k^n with respect to B_n. We use similar notations for G_m.

In this section, our coefficient ring is any commutative $\mathbb{Z}[\frac{1}{p}]$-algebra R. Therefore the only assumption we require is p being invertible in R. In particular we do not assume the existence of a square root of p in R. As the context should be clear in this section, we drop the reference to R in spaces of functions $C_c^\infty(G_k, R)$ to lighten notations. We denote by 1 the trivial representation, which should always be understood as the free R-module of rank one R with trivial group action.

2.1. **Filtration by the rank.** Let $\mathcal{O}_k$ be the set of rank k matrices in $\mathcal{M}_{n,m}(F)$ and write:

$$\mathcal{M}_{n,m}(F) = \coprod_{0 \leqslant k \leqslant m} \mathcal{O}_k$$

where each $\mathcal{O}_k$ is a single $(G_n \times G_m)$-orbit that is also a locally closed subset of $\mathcal{M}_{n,m}(F)$. Denote by $U_k = \coprod_{l \geqslant k} \mathcal{O}_l$ the set of matrices of rank at least k. The set U_k is a $(G_n \times G_m)$-stable open subset of $\mathcal{M}_{n,m}(F)$ and $\mathcal{O}_k$ is closed in U_k, yielding a stratification of the space $\mathcal{M}_{n,m}(F)$. Take representatives for $(\mathcal{O}_k)_{0 \leqslant k \leqslant m}$ by setting:

$$x_k = \begin{bmatrix} \mathrm{Id}_k & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_{n,m}(F).$$

Denote by St_k the stabiliser of x_k, which is the normal subgroup of $P_k^n \times Q_k^m$ defined by:

$$\mathrm{St}_k = \left\{ \left(\begin{bmatrix} \alpha & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ * & * \end{bmatrix} \right) \in G_n \times G_m \;\middle|\; \alpha \in G_k \right\}.$$

Write $C_c^\infty(G_k, R) \otimes_R 1$ for the representation of $P_k^n \times Q_k^m$, where the G_k factor of P_k^n acts on the left and that of Q_k^m on the right.

Proposition 2.1. *Set $\omega_{n,m}^{(k)} = C_c^\infty(U_k)$.*

a) *The rank induces a filtration in $\mathrm{Rep}_R(G_n \times G_m)$ of the Weil representation:*

$$0 \subseteq \omega_{n,m}^{(k)} \subseteq \ldots \subseteq \omega_{n,m}^{(1)} \subseteq \omega_{n,m}^{(0)} = \omega_{n,m},$$

where each subquotient is canonically isomorphic to some $C_c^\infty(\mathcal{O}_k)$.

b) *Define:*

$$W_k^{n,m} = \mathrm{ind}_{P_k^n \times Q_k^m}^{G_n \times G_m}(C_c^\infty(G_k) \otimes_R 1)$$

where the action of $P_k^n \times Q_k^m$ *is given, for* $f \in C_c^\infty(G_k)$*, by:*

$$\left(\begin{bmatrix} \alpha & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} \alpha' & 0 \\ * & * \end{bmatrix} \right) \cdot f : x \in G_k \mapsto f(\alpha^{-1} x \alpha') \in R.$$

Then the orbit map $g \in (G_n \times G_m) \mapsto g^{-1} \cdot x_k \in \mathcal{O}_k$ *factors through an homeomorphism* $\mathrm{St}_k \backslash (G_n \times G_m) \simeq \mathcal{O}_k$*, which induces canonical isomorphisms:*

$$C_c^\infty(\mathcal{O}_k) \simeq C_c^\infty(\mathrm{St}_k \backslash (G_n \times G_m)) \simeq W_k^{n,m}.$$

Proof. a) Because $\mathcal{O}_k$ is closed in U_k, we have got for all k that:

$$0 \to C_c^\infty(U_{k+1}) \overset{i_{k+1}}{\to} C_c^\infty(U_k) \overset{p_k}{\to} C_c^\infty(\mathcal{O}_k) \to 0$$

where i_{k+1} is the obvious inclusion from $U_{k+1} \subseteq U_k$ and p_k is the support restriction to $\mathcal{O}_k$. Collecting these many exact sequences for $0 \leqslant k \leqslant m$ yields the filtration.
b) First of all by [Ren09, II.3.3 Cor], the map $g \in G_n \times G_m \mapsto g^{-1} \cdot x \in \mathcal{O}_k$ induces a homeomorphism $\mathrm{St}_k \backslash (G_n \times G_m) \simeq \mathcal{O}_k$. So $C_c^\infty(\mathcal{O}_k) \simeq \mathrm{ind}_{\mathrm{St}_k}^{G_n \times G_m}(1)$ where 1 is the trivial representation. As St_k is a normal subgroup of $P_k^n \times Q_k^m$, we get that:

$$\mathrm{ind}_{\mathrm{St}_k}^{P_k^n \times Q_k^m}(1) \simeq C_c^\infty(G_k) \otimes_R 1.$$

Furthermore the action of $N_k^m \times \bar{N}_k^m$ is trivial on $C_c^\infty(G_k) \otimes_R 1$ because it is contained in St_k. So by transitivity of induction $\mathrm{ind}_{\mathrm{St}_k}^{G_n \times G_m}(1) \simeq \mathrm{ind}_{P_n^k \times Q_k^m}^{G_n \times G_m}(C_c^\infty(G_k) \otimes_R 1)$ in $\mathrm{Rep}_R(G_n \times G_m)$. □

2.2. **Example when** $n = 2$ **and** $m = 1$**.** In this situation we have a filtration:

$$0 \subseteq \omega_{2,2}^{(1)} \subseteq \omega_{2,2}^{(0)} = \omega_{2,2}.$$

Denoting by B_2 the standard Borel subgroup of G_2 of upper triangular matrices, we have:

$$W_{2,2}^1 = \mathrm{ind}_{B_2}^{G_2}(C_c^\infty(G_1) \otimes_R 1) \text{ and } W_{2,2}^0 = 1.$$

At least a part of Proposition 1.1 is already clear because we have $\mathrm{End}_{G_2 \times G_2}(1) \simeq \mathfrak{z}_R(G_0) = R$. In addition $\mathrm{Hom}_{G_2 \times G_2}(W_{2,2}^0, W_{2,2}^1) = 0$ as, for compact support reasons, there is no function in $\mathrm{ind}_{B_2}^{G_2}(C_c^\infty(G_1) \otimes_R 1)$ with support G_2. Studying the endomorphism ring of the remaining subquotient, Frobenius reciprocity reads:

$$\mathrm{End}_{G_2 \times G_1}(W_{2,2}^1) \simeq \mathrm{Hom}_{T_2 \times G_1}(\mathfrak{r}_{G_2}^{T_2}(W_{2,2}^1), C_c^\infty(G_1) \otimes_R 1)$$

and as a consequence of the geometric lemma:

$$0 \to \delta_{B_2} \cdot (1 \otimes_R C_c^\infty(G_1)) \to \mathfrak{r}_{G_2}^{T_2}(W_{2,2}^1) \to C_c^\infty(G_2) \otimes_R 1 \to 0.$$

Any morphism deduced from Frobenius reciprocity must restrict to zero on $\delta_{B_2} \cdot (1 \otimes_R C_c^\infty(G_1))$. Indeed, this is a consequence of $\mathrm{Hom}_{T_2 \times G_1}(\delta_{B_2} \cdot (1 \otimes_R C_c^\infty(G_1)), C_c^\infty(G_1) \otimes_R 1) = 0$ because, for compact support reasons again, we must have $\mathrm{Hom}_{G_1}(\chi, C_c^\infty(G_1)) = 0$. Therefore all these maps factor through the quotient in the exact sequence:

$$\begin{aligned} \mathrm{Hom}_{T_2 \times G_1}(\mathfrak{r}_{G_2}^{T_2}(W_{2,2}^1), C_c^\infty(G_1) \otimes_R 1) &\simeq \mathrm{Hom}_{T_2 \times G_1}(C_c^\infty(G_1) \otimes_R 1, C_c^\infty(G_1) \otimes_R 1) \\ &\simeq \mathrm{End}_{G_1 \times G_1}(C_c^\infty(G_1)) \simeq \mathfrak{z}_R(G_1). \end{aligned}$$

Actually these rather simple ideas (nullity of some homomorphism space for compact support reasons and using the geometric lemma) transfer well to the general case, at the cost of introducing less digestable notation.

2.3. **Representations $W_k^{n,m}$.** In order to study the properties of the Weil representation $\omega_{n,m}$, one can start considering the subquotients $W_k^{n,m}$ for $0 \leqslant k \leqslant m$. There happens that the endomorphism ring of $W_k^{n,m}$ is isomorphic to the Bernstein centre of G_k, as already noted for $n = 2$ and $m = 1$ in the previous paragraph.

Proposition 2.2. *By setting $G = G_n \times G_m$, $H = P_k^n \times Q_k^m$ and $V_k = C_c^\infty(G_k) \otimes_R 1$, the induced representation $W_k^{n,m}$ satisfies the hypothesis $\mathrm{Hom}_H(\ker(ev_1), V_k) = 0$ of Corollary A.4. In particular, we obtain an isomorphism of R-algebras:*

$$\mathrm{End}_{G_n \times G_m}(W_k^{n,m}) \overset{\curvearrowleft}{\simeq} \mathrm{End}_{G_k \times G_k}(C_c^\infty(G_k)).$$

Proof. The idea of the proof below has already been presented in the simpler case $n = 2$ and $m = 1$. There are some minor technical differences to generalise it for all n and m, but the core idea remains the same. It makes it easier to navigate through the proof below by keeping this small example in mind.

We would like to apply Corollary A.4, so we need to show that $\mathrm{Hom}_H(\ker(ev_1), V_k) = 0$. First of all, because the action of the radical unipotent of H is trivial on V_k, we deduce that:

$$\mathrm{Hom}_H(\ker(ev_1), V_k) \simeq \mathrm{Hom}_{M_k^n \times M_k^m}(\mathfrak{r}_{G_n \times G_m}^{P_k^n \times Q_k^m}(\ker(ev_1)), V_k).$$

Note that $\ker(ev_1) \subseteq \mathfrak{i}_{P_k^n \times Q_k^m}^{G_n \times G_m}(V_k)$ is the subset of functions on $G_n \times G_m$ supported on the complement of $P_k^n \times Q_k^m$. The geometric lemma as stated in Appendix A gives a filtration of $\mathfrak{r}_{G_n \times G_m}^{P_k^n \times Q_k^m}(\ker(ev_1))$ in $\mathrm{Rep}_R(M_k^n \times M_k^m)$. Its subquotients are:

$$I_{w,w'} \simeq \mathfrak{i}_{P_{(k-i,i,i)}^n \times Q_{(k-j,j,j)}^m}^{M_k^n \times M_k^m} \left(\delta_{w_{k,i}^n} \delta_{w_{k,j}^m} \otimes_R \left((w_{k,i}^n, w_{k,j}^m) \circ \mathfrak{r}_{M_k^n \times M_k^m}^{P_{(k-i,i,i)}^n \times Q_{(k-j,j,j)}^m}(V_k) \right) \right)$$

for $(w, w') = (w_{k,i}^n, w_{k,j}^m) \neq (\mathrm{Id}_n, \mathrm{Id}_m)$. In order to prove the condition $\mathrm{Hom}_H(\ker(ev_1), V_k) = 0$, it is sufficient to show that $\mathrm{Hom}_{M_k^n \times M_k^m}(I_{w,w'}, V_k) = 0$ for all $(w, w') \neq (\mathrm{Id}_n, \mathrm{Id}_m)$.

Suppose that $w \neq \mathrm{Id}_n$. Second adjunction is valid in this context [DHKM22, Cor 1.3], so the R-module $\mathrm{Hom}_{M_k^n \times M_k^m}(I_{w,w'}, V_k)$ is isomorphic to:

$$\mathrm{Hom}_{M_{(k-i,i,i)}^n \times M_k^m} \left(\mathfrak{i}_{Q_{(k-j,j,j)}^m}^{M_k^m} \left(\delta_{(w_{k,i}^n, w_{k,j}^m)} \otimes_R \left((w_{k,i}^n, w_{k,j}^m) \circ \cdots \right) \right), \bar{\mathfrak{r}}_{M_k^n}^{P_{(k-i,i,i)}^n}(V_k) \right).$$

Because $w \neq \mathrm{Id}_n$, we have $i \neq 0$. Consider the following non-trivial torus:

$$T_i^n = \begin{bmatrix} T_i & \\ & \mathrm{Id}_{n-k} \end{bmatrix}, \text{ where } T_i = \left\{ \begin{bmatrix} \mathrm{Id}_{k-i} & \\ & \lambda \mathrm{Id}_i \end{bmatrix} \in G_k \;\middle|\; \lambda \in F^\times \right\}.$$

Then we claim that T_i^n acts as a character on the left-hand side of the last Hom-space above whereas it can not act as a character on the right-hand side. Indeed we have:

$$\bar{\mathfrak{r}}_{M_k^n}^{P_{(k-i,i,i)}^n}(V_k) = \mathfrak{r}_{M_k^n}^{Q_{(k-i,i,i)}^n}(V_k) \simeq C_c^\infty(\bar{N}_i^k \backslash G_k) \otimes_R 1.$$

If T_i^n acts as a character on $v \in C_c^\infty(\bar{N}_i^k \backslash G_k) \otimes_R 1$, then T_i acts a character on some element $v' \in C_c^\infty(\bar{N}_i^k \backslash G_k)$. But if $v' \neq 0$ then its support must contain $\bar{N}_i^k T_i$, which is not compact in $\bar{N}_i^k \backslash G_k$, so v' must be zero *i.e.* $v = 0$. This means that the Hom-space above must be zero too as $T_i^n \subseteq M_{(k-i,i,i)}^n$.

Therefore $\mathrm{Hom}_{M_k^n \times M_k^m}(I_{w,w'}, V_k) = 0$ for all (w, w') with $w \neq \mathrm{Id}_n$. Alternatively we can conclude this is also zero for all $(w, w') \neq (\mathrm{Id}_n, \mathrm{Id}_m)$ when $w' \neq \mathrm{Id}_m$, just switching the roles of w and w' in the proof above. Therefore Corollary A.4 applies as $\ker(\mathrm{ev}_1)$ has a filtration whose subquotients are $(I_{w,w'})$ for $(w, w') \neq (\mathrm{Id}_n, \mathrm{Id}_m)$. □

With similar arguments to the proof right above, we can prove:

Proposition 2.3. *For all $k' > k$, we have $\mathrm{Hom}_{G_n \times G_m}(W_k^{n,m}, W_{k'}^{n,m}) = 0$.*

Proof. As this proof is just a mere variation of the previous one, we go through the arguments in a more direct way. Set $V_l = C_c^\infty(G_l) \otimes_R 1 \in \mathrm{Rep}_R(P_l^n \times Q_l^m)$. By Frobenius reciprocity:

$$\mathrm{Hom}_{G_n \times G_m}(W_k^{n,m}, W_{k'}^{n,m}) \simeq \mathrm{Hom}_{M_{k'}^n \times G_m}(\mathfrak{r}_{G_n}^{P_{k'}^n} \circ i_{P_k^n}^{G_n}(\mathfrak{i}_{Q_k^m}^{G_m}(V_k)), \mathfrak{i}_{Q_k^n}^{G_m}(V_{k'})).$$

Here the version of the geometric lemma we use is again exposed in Appendix A, where the index set is $W(k,k',n)$ and its elements are $w_i = w_{k,k',i}^n \in W(k,k',n)$ for $i \in [\![0, \min(k, n-k')]\!]$. To have lighter formulas set $V_l^m = \mathfrak{i}_{Q_l^m}^{G_m}(V_l^m) \in \mathrm{Rep}_R(P_l^n \times G_m)$. The subquotients read:

$$I_{w_i} = \mathfrak{i}_{M_{(k-i,k'-k+i,i)}^n}^{M_{k'}^n}\left(\delta_{w_i} \otimes_R (w_i \circ \mathfrak{r}_{M_k^n}^{M_{(k-i,i,k'-k+i)}^n}(V_k^m)\right)$$

and we want to prove that $\mathrm{Hom}_{M_{k'}^n \times G_m}(I_{w_i}, V_{k'}^m) = 0$ for all $w_i \in W(k,k',n)$.

Applying the second adjunction [DHKM22, Cor 1.3], we want to prove that:

$$\mathrm{Hom}_{M_{(k-i,k'-k+i,i)}^n \times G_m}(\delta_{w_i} \otimes_R (w_i \circ \mathfrak{r}_{M_k^n}^{M_{(k-i,i,k'-k+i)}^n}(V_k^m)), \bar{\mathfrak{r}}_{M_{k'}^n}^{M_{(k-i,k'-k+i,i)}^n}(V_{k'}^m)) = 0.$$

Similarly to the previous proof, the non-trivial torus:

$$\left\{ \begin{bmatrix} \mathrm{Id}_k & & \\ & \lambda \mathrm{Id}_{k'-k} & \\ & & \mathrm{Id}_{n-k'} \end{bmatrix} \in G_n \;\middle|\; \lambda \in F^\times \right\}$$

acts as a character on the left-hand side of the Hom-space, but it can not act as a character on the right-hand side $\bar{\mathfrak{r}}_{M_{k'}^n}^{M_{(k-i,k'-k+i,i)}^n}(V_{k'}^m) \simeq C_c^\infty(\bar{N}_{k-i}^{k'} \backslash G_{k'}) \otimes_R 1$ for compact support reasons. Therefore $\mathrm{Hom}_{M_{k'}^n \times G_m}(I_{w_i}, V_{k'}^m) = 0$ for all w_i. □

3. Proof of Proposition 1.2

By abuse of notation, we write $\varphi_{n,m}(z)$ for $z \in Z_A(G_n) \otimes Z_A(G_m)$ in matrix representation with respect to the filtration of the Weil representation. As the action of the center preserve subrepresentations, we must have:

$$\varphi_{n,m}(z) = \begin{bmatrix} \varphi_{n,m}^m(z) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \varphi_{n,m}^0(z) \end{bmatrix}.$$

In addition, we can consider the $*$ maps above to be 0 because of Proposition 2.3 and this implies the following relation on kernels:

$$\mathrm{Ker}(\varphi_{n,m}) = \bigcap_{k=0}^{m} \mathrm{Ker}(\varphi_{n,m}^k).$$

We now want to prove Proposition 1.2 in a special case:

Proposition 3.1. *Let $A = \mathbb{Z}[1/p]$. We have inclusions of kernels:*

$$\mathrm{Ker}(\varphi_{n,m}^m) \subseteq \ldots \subseteq \mathrm{Ker}(\varphi_{n,m}^1) \subseteq \mathrm{Ker}(\varphi_{n,m}^0).$$

Proof. Let $z \in Z_{\mathbb{Z}[1/p]}(G_n) \otimes Z_{\mathbb{Z}[1/p]}(G_m)$ such that $\varphi_{n,m}^{k+1}(z) = 0$, then the goal is $\varphi_{n,m}^k(z) = 0$. Because $\varphi_{n,m}^k(z)$ belongs to $Z_{\mathbb{Z}[1/p]}(G_k)$, it is enough to check it over the complex numbers via the canonical embedding:

$$Z_{\mathbb{Z}[1/p]}(G_k) \hookrightarrow Z_{\mathbb{C}}(G_k)$$

and the compatibility of the Weil representation with scalar extension.

Consider the subquotients $W_{n,m}^k$ of the filtration of Proposition 2.1 when $R = \mathbb{C}$. Applying Lemma B.1 to each direct factor of the center, we have $\varphi_{n,m}^k(z) = 0$ if and only if for any Zariski open dense subset U we have $\eta(\varphi_{n,m}^k(z)) = 0$ for all $\eta : Z_{\mathbb{C}}(G_k) \to \mathbb{C}$ in U. Then combining it with Corollary B.2 and Proposition B.5, we are left to check $\eta(\varphi^k_{n,m}(z)) = 0$ where this scalar is the action of z on:

$$(W^k_{n,m})_\eta = \mathrm{Ind}_{P^n_k}^{G_n}(\pi_\eta \otimes 1_{n-k}) \otimes_{\mathbb{C}} \mathrm{Ind}_{Q^m_k}^{G_m}(\pi^\vee_\eta \otimes 1_{m-k}).$$

In order to introduce a term coming from $\mathrm{Ind}_{P_{k+1}}^{G_n}$ and use our hypothesis $\varphi^{k+1}_{n,m}(z) = 0$, we use induction in stages. First, embed:

$$1_{n-k} \subseteq \mathrm{Ind}_{P_1^{n-k}}^{G_{n-k}}(1_1 \otimes 1_{n-(k+1)}).$$

Similarly for m. Remark that the induced representation above has always length 2 by the theory of segments: it contains 1_{n-k} as well as an irreducible quotient σ. Now by transitivity of induction we can embed $(W^k_{n,m})_\eta$ in:

$$\mathrm{Ind}_{P^n_{k+1}}^{G_n}(\mathrm{Ind}_{P_k^{k+1}}^{G_{k+1}}(\pi_\eta \otimes 1_1) \otimes 1_{n-(k+1)}) \otimes \mathrm{Ind}_{Q^m_{k+1}}^{G_m}(\mathrm{Ind}_{Q_k^{k+1}}^{G_{k+1}}(\pi^\vee_\eta \otimes 1_1) \otimes 1_{m-(k+1)}). \tag{1}$$

Thanks to the theory of segments, we can always assume for all η in our Zariski dense open set U that $\mathrm{Ind}_{P_k^{k+1}}^{G_{k+1}}(\pi_\eta \otimes 1_1)$ and $\mathrm{Ind}_{Q_k^{k+1}}^{G_{k+1}}(\pi^\vee_\eta \otimes 1_1)$ are irreducible. Therefore it defines an irreducible quotient of the regular representation:

$$C^\infty_c(G_{k+1}) \twoheadrightarrow \mathrm{Ind}_{P_k^{k+1}}^{G_{k+1}}(\pi_\eta \otimes 1_1) \otimes \mathrm{Ind}_{P_k^{k+1}}^{G_{k+1}}(\pi_\eta \otimes 1_1)^\vee$$

which induces a quotient of $W^{k+1}_{n,m} = \mathrm{Ind}_{P^n_{k+1} \times Q^m_{k+1}}^{G_n \times G_m}(C^\infty_c(G_{k+1}) \otimes 1)$, namely:

$$\mathrm{Ind}_{P^n_{k+1}}^{G_n}(\mathrm{Ind}_{P_k^{k+1}}^{G_{k+1}}(\pi_\eta \otimes 1_1) \otimes 1_{n-(k+1)}) \otimes \mathrm{Ind}_{Q^m_{k+1}}^{G_m}(\mathrm{Ind}_{P_k^{k+1}}^{G_{k+1}}(\pi_\eta \otimes 1_1)^\vee \otimes 1_{m-(k+1)}). \tag{2}$$

Note that z acts as $\varphi^{k+1}_{n,m}(z) = 0$ on $W^{k+1}_{n,m}$, so it acts trivially on the latter quotient (2) of $W^{k+1}_{n,m}$. We will now relate this quotient to our big tensor product (1) above.

It remains to check that the right-hand side of the tensor products (1) and (2) define the same supercuspidal support. It is easier to switch to normalized induction to read supercuspidal support as with the non-normalized version, one needs to introduce twists by modulus characters. Choosing a square root of q in $\mathbb{C}$, we only need to compare the supercuspidal supports of:

$$\mathrm{Ind}_{Q_k^{k+1}}^{G_{k+1}}(\pi^\vee_\eta \otimes 1_1) = i_{Q_k^{k+1}}^{G_{k+1}}(\delta^{-\frac{1}{2}}_{Q_k^{k+1}}(\pi^\vee_\eta \otimes 1)) \text{ and } \mathrm{Ind}_{P_k^{k+1}}^{G_{k+1}}(\pi_\eta \otimes 1_1)^\vee = i_{P_k^{k+1}}^{G_{k+1}}(\delta^{\frac{1}{2}}_{P_k^{k+1}}(\pi^\vee_\eta \otimes 1_1)).$$

But $\delta_{Q_k^{k+1}} = \delta^{-1}_{P_k^{k+1}}$ as Q_k^{k+1} is the opposite parabolic of P_k^{k+1}. So the superscuspidal supports are the same. To sum up what we have obtained: the scalar $\eta(\varphi^k_{n,m}(z))$ corresponds to the action of z on $(W^k_{n,m})_\eta$ and it is also equal to the scalar $\eta'(\varphi^{k+1}_{n,m}(z))$ corresponding to the action of z on $(W^{k+1}_{n,m})_{\eta'}$ where $\eta' : Z_{\mathbb{C}}(G_{k+1}) \to \mathbb{C}$ is the supercupsidal support associated to the irreducible representation $\mathrm{Ind}_{Q_k^{k+1}}^{G_{k+1}}(\pi^\vee_\eta \otimes 1)$. □

Corollary 3.2. *Let A be any $\mathbb{Z}[1/p]$-algebra. We have an inclusion of kernels:*

$$\mathrm{Ker}(\varphi^m_{n,m}) \subseteq \ldots \subseteq \mathrm{Ker}(\varphi^1_{n,m}) \subseteq \mathrm{Ker}(\varphi^0_{n,m}).$$

Proof. The A-algebra $Z_A(G_n) \otimes Z_A(G_m)$ is generated by the image of $Z_{\mathbb{Z}[1/p]}(G_n) \otimes Z_{\mathbb{Z}[1/p]}(G_m)$ thanks to Corollary B.11. As the maps $\varphi^k_{n,m}$ are compatible to scalar extension, and so does the Weil representation, the statement follows from the fact that it holds over $\mathbb{Z}[1/p]$. □

We deduce from the previous corollary the existence of our morphism:

Proposition 3.3. *There exists a unique $\mathbb{Z}[1/p]$-algebra morphism:*

$$\theta^\#_{n,m} : \mathcal{Z}_{\mathbb{Z}[1/p]}(G_n) \to \mathcal{Z}_{\mathbb{Z}[1/p]}(G_m)$$

such that for all $z \in \mathcal{Z}_{\mathbb{Z}[1/p]}(G_n)$ we have $\varphi^m_{n,m}(z \otimes 1) = \varphi^m_{n,m}(1 \otimes \theta^\#_{n,m}(z))$. Furthermore this construction is compatible to scalar extension to any $\mathbb{Z}[1/p]$-algebra A.

Proof. First of all, assuming the map $\theta^{\#}_{n,m}$ exists, the second part holds thanks to the compatibility to scalar extension of the Weil representation together with Corollary B.11. Looking at the definition of $\varphi^m_{n,m}$, this map is uniquely determined by the action of $\mathcal{Z}_{\mathbb{Z}[1/p]}(G_n)$ on $W^m_{n,m}$ and the canonical identification $\mathrm{End}_{G_n\times G_m}(W^m_{n,m}) \simeq \mathcal{Z}_{\mathbb{Z}[1/p]}(G_m)$ from Proposition 2.2. □

4. Theta correspondence and supercuspidal support

Let R be an algebraically closed field of characeristic $\ell \neq p$. Let $\omega_{m,n}$ be the Weil representation with coefficients in R for the group $G_m \times G_n$ and assume that $n \geqslant m$.

4.1. **Banal theta correspondence.** Recall that if the characteristic ℓ of R does not divide the pro-orders of G_m and G_n, we say that ℓ is banal with respect to G_n and G_m. The following theorem constitutes the heart of the theta correspondence and has been proved by Roger Howe for complex coefficients $R = \mathbb{C}$ and by Alberto Mínguez for any algebraically closed field R of banal characteristic with respect to G_n and G_m:

Theorem 4.1 (Howe, Mínguez)**.** *Let $\pi \in \mathrm{Irr}_R(G_m)$. Then:*

(i) there exists a unique $\theta(\pi) \in \mathrm{Irr}_R(G_n)$ such that $\omega_{m,n} \twoheadrightarrow \pi \otimes_R \theta(\pi)$;
(ii) the map $\pi \mapsto \theta(\pi)$ thus defined is injective;
(iii) the quotient is multiplicity one i.e. $\dim_{G_n\times G_m}(\omega_{m,n}, \pi \otimes_R \theta(\pi)) = 1$.

Write $\mathrm{Irr}^\theta_R(G_n)$ to denote the image of the map θ. Then the theorem asserts a bijection:

$$\mathrm{Irr}_R(G_m) \overset{\theta}{\simeq} \mathrm{Irr}^\theta_R(G_n).$$

The map θ of the theorem is reputed to be compatible with the supercuspidal support, which means there exists a map $\theta_{\mathrm{scs}} : \Omega_{\mathrm{scs}}(G_m) \to \Omega_{\mathrm{scs}}(G_n)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Irr}_R(G_m) & \xrightarrow{\theta} & \mathrm{Irr}_R(G_n) \\ \downarrow & & \downarrow \\ \Omega_{\mathrm{scs}}(G_m) & \xrightarrow{\theta_{\mathrm{scs}}} & \Omega_{\mathrm{scs}}(G) \end{array} .$$

Similarly to the map θ, we can denote by $\Omega^\theta_{\mathrm{scs}}(G)$ the image of θ_{scs}, which alternatively is the image of $\mathrm{Irr}^\theta_R(G_n)$ through the supercuspidal support. It is not *a priori* clear whether the map θ_{scs} thus defined is injective or not. However this map θ_{scs} is injective because it can be described in the following way:

$$\begin{array}{ccc} \tilde{\Omega}_R(G_m) & \to & \tilde{\Omega}_R(G_n) \\ (M,\rho) & \mapsto & (M \times T_{n-m}, \rho^\vee \chi_M \otimes_R \chi_{T_{n-m}}) \end{array}$$

where $\tilde{\Omega}_R(G_m)$ is the set of supercuspidal pairs (M,ρ) where M is a Levi and ρ is a supercuspidal representation of this Levi and the characters of M and T_{n-m} respectively are:

$$\chi_M = |\cdot|^{-\frac{n-m}{2}} \text{ and } \chi_{T_{n-m}} = |\cdot|_1^{(m+1-n)+\frac{(n-1)}{2}} \otimes_R \cdots \otimes_R |\cdot|_1^{\frac{(n-1)}{2}} .$$

This map is compatible with the equivalence relation on each side, also called the association class of supercuspidal pairs, and therefore defines a map:

$$\begin{array}{ccc} \Omega_R(G_m) & \to & \Omega_R(G_n) \\ (M,\rho)_{\mathrm{scs}} & \mapsto & (M \times T_{n-m}, \rho^\vee \chi_M \otimes_R \chi_{T_{n-m}})_{\mathrm{scs}} \end{array}$$

where we denote equivalence classes by subscripts. This map is precisely θ_R and one can easily check that the explicit latter map is injective. So θ_R induces a bijection $\Omega_R(G_m) \simeq \Omega^\theta_R(G_n)$.

4.2. **Non-banal situation.** In the non-banal setting, the point (1) of Theorem 4.1 already fails. Indeed when ℓ divides $q^n - 1$, there is a counter-example due to Mínguez considering the restriction to $\{0\} \subseteq \mathcal{M}_{n,1}(D)$ and a Haar measure of $\mathcal{M}_{n,1}(D)$ to obtain a surjection:

$$\omega_{1,n} \twoheadrightarrow (1 \otimes_R 1_n) \oplus_R (1 \otimes_R |\cdot|_n)$$

where 1_n is the trivial representation of G_n and $|\cdot|_n = |\cdot|_F \circ \det$ is the norm character of G_n. As opposed to the banal setting, these two characters 1_n and $|\cdot|_n$ share the same supercuspidal support. Even though defining a bijective map θ_R in terms of irreducible representatons will fail in the non-banal setting, this counter-example suggests that the supercuspidal support could be preserved by a good candidate like:

$$\begin{array}{rccc} \theta_R : & \Omega_R(G_m) & \to & \Omega_R(G_n) \\ & (M,\rho)_{\mathrm{scs}} & \mapsto & (M \times T_{n-m}, \rho^\vee \chi_M \otimes_R \chi_{T_{n-m}})_{\mathrm{scs}} \end{array}.$$

4.3. **Map of varieties.** Another way to think about supercuspidal supports is to identify them with the points of the Bernstein centre. For all irreducible $\pi \in \mathrm{Irr}_R(G)$, the Bernstein centre is acting as a character $\eta_\pi : z \in \mathcal{Z}_R(G) \mapsto z_\pi \in R$ thanks to Schur's lemma. This is a result of Vignéras that the equivalence relation on $\mathrm{Irr}_R(G)$ defined by "having the same character" agrees with supercuspidal support *i.e.* $\eta_\pi = \eta_{\pi'}$ if and only if $\mathrm{scs}(\pi) = \mathrm{scs}(\pi')$. For $\mathfrak{m}$ a supercuspidal support, we denote by $\eta_{\mathfrak{m}}$ the associated character. Therefore we have got a bijection:

$$\begin{array}{ccc} \Omega_R(G) & \to & \mathrm{Hom}_{R-\mathrm{alg}}(\mathcal{Z}_R(G), R) \\ \mathfrak{m} & \mapsto & \eta_{\mathfrak{m}} \end{array}.$$

In particular $\Omega_R(G)$ can be endowed with a structure of affine scheme as $\Omega_R(G)$ is naturally identified with $\mathrm{Spec}(\mathcal{Z}_R(G))(R)$. When ℓ is banal with respect to G, one can describe the irreducible components of $\mathcal{Z}_R(G)$, which alternatively correspond to the set of primitive idempotents in $\mathcal{Z}_R(G)$. Each one of these irreducible components is also connected and finite type over R.

With this point of view, one can ask about the algebraicity of the map θ_R defined above:

Proposition 4.2. *Let ℓ be banal with respect to G_n and G_m. The map θ_R induces a morphism of algebraic varieties $X_{m,R} \to X_{n,R}$.*

Proof. We decompose the map θ_R as the composition of:

$$(M,\rho)_{\mathrm{scs}} \mapsto (M,\rho^\vee)_{\mathrm{scs}} \text{ and } (M,\rho)_{\mathrm{scs}} \mapsto (M \times T_{n-m}, \rho\chi_M \otimes_R \chi_{T_{n-m}})_{\mathrm{scs}}.$$

The first one is algebraic as it corresponds to the contragredient involution on the centre, which is an automorphism of $\mathcal{Z}_R(G_m)$.

Regarding the second one, let $\mathfrak{s} \in \mathcal{B}_R(G_m)$ be an inertial support and let (M,σ) be a supercuspidal pair such that $(M,\sigma)_{\mathrm{scs}} \in \mathfrak{s}$. Let $X_R(M)$ be the variety of unramified characters for the Levi M. The map:

$$\psi \in X_R(M) \mapsto (M, \sigma\psi) \in \Omega_R^{\mathfrak{s}}(G_m)$$

identifies $\Omega_R^{\mathfrak{s}}(G_m)$ with the quotient $X_R(M)/H_{(M,\sigma)}$ where $H_{(M,\sigma)}$ is the finite group corresponding to all characters $\psi \in X_R(M)$ such that $(M,\sigma\psi)_{\mathrm{scs}} = (M,\sigma)_{\mathrm{scs}}$. We have this relation if we can find $w \in N_{G_m}(M)/M$ such that $(\sigma\psi)^w \simeq \sigma\psi$. Similarly let $\mathfrak{s}' \in \mathcal{B}_R(G_n)$ such that $(M \times T_{n-m}, \sigma\chi_M \otimes_R \chi_{T_{n-m}})_{\mathrm{scs}} \in \mathfrak{s}'$ and identify the variety $\Omega_R^{\mathfrak{s}'}(G_n)$ with the quotient $X_R(M \times T_{n-m})/H_{(M \times T_{n-m}, \sigma\chi_M \otimes_R \chi_{T_{n-m}})}$. We can define the algebraic map:

$$(M,\sigma\psi) \mapsto (M \times T_{n-m}, \sigma\psi\chi_M \otimes_R \chi_{T_{n-m}}).$$

In order for the map:

$$(M,\rho)_{\mathrm{scs}} \mapsto (M \times T_{n-m}, \rho\chi_M \otimes_R \chi_{T_{n-m}})_{\mathrm{scs}}$$

to be algebraic it is sufficient to check whether the algebraic map:

$$(M,\sigma\psi) \mapsto (M \times T_{n-m}, \sigma\chi_M \otimes_R \chi_{T_{n-m}})$$

induces a map $\Omega_R^{\mathfrak{s}}(G_m) \to \Omega_R^{\mathfrak{s}'}(G_n)$ on quotients of $X_R(M)$ and $X_R(M \times T_{n-m})$. But for all $\psi \in H_{(M,\sigma)}$ we claim that:

$$(M \times T_{n-m}, \sigma\psi\chi_M \otimes_R \chi_{T_{n-m}})_{\mathrm{scs}} = (M \times T_{n-m}, \sigma\chi_M \otimes_R \chi_{T_{n-m}})_{\mathrm{scs}}.$$

Indeed our map is equivariant for:

$$w \in N_{G_m}(M)/M \to (w, \mathrm{Id}_{T_{n-m}}) \in N_{G_n}(M \times T_{n-m})/(M \times T_{n-m})$$

in the sense that:

$$(\sigma\psi\chi_M \otimes_R \chi_{T_{n-m}})^{(w,\mathrm{Id}_{T_{n-m}})} \simeq (\sigma\psi\chi_M)^w \otimes_R \chi_{T_{n-m}} \simeq (\sigma\psi)^w \chi_M \otimes_R \chi_{T_{n-m}}$$

where we have used $\chi_M^w = \chi_M$ for all $w \in N_{G_m}(M)/M$ because χ_M factors through the determinant over M. Therefore we obtain a map $\Omega_R^{\mathfrak{s}}(G_m) \to \Omega_R^{\mathfrak{s}'}(G_n)$. □

The algebraic morphism θ_R is defined on coordinate rings by a single $\theta_R^{\#} : \mathcal{Z}_R(G_n) \to \mathcal{Z}_R(G_m)$ because the ring $\mathcal{Z}_R(G_m)$ is reduced according to our banality assumption. Between principal blocks this map has already been studied by Rallis who studied:

$$(T_m, \psi) \mapsto (T_n, \psi| \cdot |_m^{-\frac{n-m}{2}} \otimes_R | \cdot |_1^{(m+1-n)+\frac{(n-1)}{2}} \otimes_R \cdots \otimes_R | \cdot |_1^{\frac{(n-1)}{2}})$$

and which corresponds writing $T_n = T_m \times T_{n-m}$ to the R-algebra morphism:

$$(\theta_R^{(T_m,1_m)})^{\#} : R[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \mapsto R[X_1^{\pm 1}, \cdots, X_m^{\pm 1}]$$

sending X_i on $q^{-\frac{n-m}{2}} X_i^{-1}$ if $1 \leqslant i \leqslant m$ and $q^{(i-n)+\frac{n-1}{2}}$ if $m+1 \leqslant i \leqslant n$. This map is compatible with the permutation action on variables from $\mathfrak{S}_n$ and $\mathfrak{S}_m$ in the sense that there is a commutative diagram:

$$\begin{array}{ccc} R[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] & \xrightarrow{(\theta_R^{(T_m,1_m)})^{\#}} & R[X_1^{\pm 1}, \cdots, X_m^{\pm 1}] \\ \uparrow & & \uparrow \\ R[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]^{\mathfrak{S}_n} & \xrightarrow{(\theta_R^{(T_m,1_m)_{\mathrm{scs}}})^{\#}} & R[X_1^{\pm 1}, \cdots, X_m^{\pm 1}]^{\mathfrak{S}_m} \end{array} .$$

It can be easily checked that the R-algebra morphism obtained on invariants, which we refer to as Rallis' map, is surjective. This extends basically thanks to the block decomposition:

Proposition 4.3. *Under the banal assumption, the map $\theta_R^{\#} : \mathcal{Z}_R(G_n) \to \mathcal{Z}_R(G_m)$ is surjective. So θ_R is a closed immersion.*

Proof. We can explicitly write what the map θ_R on Bernstein components. First of all, note that the morphism $a \in \mathbb{G}_{m,R} \to \lambda a^{-1} \in \mathbb{G}_{m,R}$ corresponds to the morphism of R-algebras $X \in R[X^{\pm 1}] \mapsto \lambda X^{-1} \in R[X^{\pm 1}]$. Recall the notations from the previous proof where (M, σ) is a supercuspidal pair and $\mathfrak{s} \in \mathcal{B}_R(G_m)$ such that $(M, \sigma)_{\mathrm{scs}} \in \mathfrak{s}$, as well as the image $\mathfrak{s}' \in \mathcal{B}_R(G_n)$. We choose the supercuspidal $(M \times T_{n-m}, \sigma \otimes_R 1_{n-m})_{\mathrm{scs}} \in \mathfrak{s}'$ as our base point in $\Omega_R^{\mathfrak{s}'}(G_n)$. We can also assume that (M, σ) is factored according to its principal unramified part in the sense that $(M, \sigma) = (M_0 \times T_k, \sigma_0 \otimes 1_k)$ where σ_0 does not contain any unramified character of a torus.

Consider the R-algebra morphism:

$$\bar{m} \otimes_R \bar{t}_{n-m} \in R[M/M^0] \otimes_R R[T_{n-m}/T_{n-m}^0] \mapsto \chi_{T_{n-m}}(\bar{t}_{n-m})\chi_M(\bar{m})\bar{m}^{-1} \in R[M/M^0]$$

which corresponds to $(M, \sigma\psi) \mapsto (M, \sigma^\vee \psi^{-1} \chi_M \otimes_R \chi_{T_{n-m}})$. Thanks to the identification $T_{k+n-m} = T_k \times T_{n-m}$ through $t_{n-m+k} = (t_k, t_{n-m})$ where k was our principal unramified index, we can rewrite it as $(M_0 \times T_k, \sigma_0 \psi_{M_0} \otimes_R \psi_k) \mapsto (M_0 \times T_{k+n-m}, \sigma^\vee \psi_0^{-1} \chi_{M_0} \otimes_R \chi_{T_{k+n-m}} \psi_k^{-1})$ for:

$$\chi_{M_0} = \chi_M|_{M_0} \text{ and } \chi_{T_{k+n-m}} = \chi_M|_{T_k} \chi_{T_{n-m}} = | \cdot |_{T_k}^{-\frac{n-m}{2}} \otimes_R | \cdot |_1^{(m+1-n)+\frac{(n-1)}{2}} \otimes_R \cdots \otimes_R | \cdot |_1^{\frac{(n-1)}{2}}.$$

This corresponds on $R[M_0/M_0^0] \otimes_R R[T_{k+n-m}/T_{k+n-m}^0] \to R[M_0/M_0^0] \otimes_R R[T_k/T_k^0]$ to:

$$(\theta_R^{(M,\sigma)})^{\#} : \bar{m}_0 \otimes_R \bar{t}_{n-m+k} \mapsto \chi_{M_0}(\bar{m}_0)\bar{m}_0^{-1} \otimes_R \chi_{T_{n-m+k}}(\bar{t}_{n-m+k})\bar{t}_k^{-1}.$$

The latter map is equivariant for the action of the groups:

$$H_{(M \times T_{n-m}, \sigma^\vee \otimes_R 1_{n-m})} = H_{(M_0,\sigma_0)} \times H_{(T_{k+n-m}, 1_{k+n-m})} = H_{(M_0,\sigma_0)} \times \mathfrak{S}_{k+n-m}$$

and its subgroup:

$$H_{(M,\sigma)} = H_{(M_0,\sigma_0)} \times \mathfrak{S}_k$$

obtained from the embedding $\mathfrak{S}_k = N_{G_k}(T_k)/T_k \subseteq \mathfrak{S}_{k+n-m} = N_{G_{k+n-m}}(T_{k+n-m})/T_{k+n-m}$. The map $(\theta_R^{(M,\sigma)})^\#$ is compatible to the action of these groups and induces a R-algebra morphism $(\theta_R^{(M,\sigma)_{\mathrm{scs}}})^\#$ for invariant subrings:

$$\begin{array}{ccc} R[M_0/M_0^0] \otimes_R R[T_{k+n-m}/T^0_{k+n_m}] & \longrightarrow & R[M_0/M_0^0] \otimes_R R[T_k/T_k^0] \\ \uparrow & & \uparrow \\ R[M_0/M_0^0]^{H_{(M_0,\sigma_0)}} \otimes_R R[T_{k+n-m}/T^0_{k+n_m}]^{\mathfrak{S}_{k+n-m}} & \longrightarrow & R[M_0/M_0^0]^{H_{(M_0,\sigma_0)}} \otimes_R R[T_k/T_k^0]^{\mathfrak{S}_k} \end{array} .$$

Here $R[M_0/M_0^0]^{H_{(M_0,\sigma_0)}} \to R[M_0/M_0^0]^{H_{(M_0,\sigma_0)}}$ is an isomorphism induced by $\bar{m}_0 \mapsto \chi_{M_0}(\bar{m}_0)\bar{m}_0^{-1}$ whereas $R[T_{k+n-m}/T^0_{k+n_m}]^{\mathfrak{S}_{k+n-m}} \to R[T_k/T_k^0]^{\mathfrak{S}_k}$ is Rallis' map. □

When we remove the banal assumption, the scalar extension of $\theta^\#_{n,m}$ to R still provides an R-algebra morphism $\mathcal{Z}_R(G_n) \to \mathcal{Z}_R(G_m)$ but it seems difficult to determine whether or not it is surjective. The main difficulty comes from the fact that our Bernstein centre rings are no longer reduced and therefore the map θ_R on supercuspidal supports does not come from a single ring morphism. Moreover the description of the connected components of the centre as some ring of invariants is only known in the banal setting. Nevertheless we are going to prove that this morphism is finite.

An other question we do not address but that should be easier to answer after the forthcoming work [DHKM23] is whether, after inverting the radical N of all the non-banal primes for the groups G_n and G_m, the map induced by scalar extension $\mathcal{Z}_{\mathbb{Z}[1/pN]}(G_n) \to \mathcal{Z}_{\mathbb{Z}[1/pN]}(G_m)$ is surjective.

5. Other applications

5.1. **Finiteness of $\theta^\#_{n,m}$.** In Proposition 3.3, we obtained the morphism of $\mathbb{Z}[1/p]$-algebras $\theta^\#_{n,m}$ by considering the natural action of $\mathcal{Z}_{\mathbb{Z}[1/p]}(G_n)$ on $W^m_{n,m} = \mathrm{ind}_{P^n_m}^{G_n}(C_c^\infty(G_m) \otimes 1)$. This situation can be translated in terms of Harish-Chandra morphisms to prove a finiteness statement:

Lemma 5.1. *Let $\sigma_m = C_c^\infty(G_m) \otimes 1 \in \mathrm{Rep}_{\mathbb{Z}[1/p]}(M^n_m \times G_m)$ as in Proposition 2.1. Then for all $z \in \mathcal{Z}_{\mathbb{Z}[1/p]}(G_n)$ we have:*

$$z_{W^m_{n,m}} = \mathrm{ind}_{P^n_m}^{G_n}(HC(z)_{\sigma_m})$$

where $HC : \mathcal{Z}_{\mathbb{Z}[1/p]}(G_n) \to \mathcal{Z}_{\mathbb{Z}[1/p]}(M^n_m)$ is the Harish-Chandra morphism.

Proof. We have $W^m_{n,m} = \mathrm{ind}_{P^n_m}^{G_n}(C_c^\infty(G_m) \otimes 1)$ and we can apply [DHKM22, Th 4.1]. □

Proposition 5.2. *There exists a surjective map $\gamma_{\sigma_m} : \mathcal{Z}_{\mathbb{Z}[1/p]}(M^n_m) \twoheadrightarrow \mathcal{Z}_{\mathbb{Z}[1/p]}(G_m)$ such that:*

$$\theta^\#_{n,m} = \gamma_{\sigma_m} \circ HC.$$

Proof. We identify σ_m and $C_c^\infty(G_k)$ in an obvious way, so $HC(z)_{\sigma_m} \in \mathrm{End}_{G_m \times G_m}(C_c^\infty(G_m))$. Let ρ_l and ρ_r be respectively the left and the right action on the regular representation $C_c^\infty(G_k)$. For $z \in \mathcal{Z}_{\mathbb{Z}[1/p]}(G_n)$, we have:

$$\rho_r(\theta^\#_{n,m}(z)) = \rho_l(\theta^\#_{n,m}(z)^\vee) \in \mathrm{End}_{G_k \times G_k}(C_c^\infty(G_k)).$$

We can take γ_{σ_m} to be the composition of $\mathrm{ev}_{\sigma_m} : \mathcal{Z}_{\mathbb{Z}[1/p]}(M^n_m) \to \mathrm{End}_{G_k \times G_k}(C_c^\infty(G_k))$, given by the action of $\mathcal{Z}_{\mathbb{Z}[1/p]}(M^n_m)$ on σ_m, with the isomorphism $\mathrm{End}_{G_k \times G_k}(C_c^\infty(G_k)) \to \mathcal{Z}_{\mathbb{Z}[1/p]}(G_m)$ that is the inverse of $z \in \mathcal{Z}_{\mathbb{Z}[1/p]}(G_m) \mapsto \rho_l(z) \in \mathrm{End}_{G_k \times G_k}(C_c^\infty(G_k))$. Note that the morphism ev_{σ_m} is surjective because $\mathcal{Z}_{\mathbb{Z}[1/p]}(M^n_m) \simeq \mathcal{Z}_{\mathbb{Z}[1/p]}(G_m) \otimes \mathcal{Z}_{\mathbb{Z}[1/p]}(G_{n-m})$. So the map γ_{σ_m} is surjective and the equality of the proposition holds. □

By the compatibility to scalar extension of our map $\theta^\#_{n,m}$ and the finiteness of Harish-Chandra morphisms [DHKM22, Th 4.3], we obtain:

Corollary 5.3. *Let A be a noetherian $\mathbb{Z}_\ell$-algebra. Then $\theta^{\#}_{n,m} : \mathcal{Z}_A(G_n) \to \mathcal{Z}_A(G_n)$ is finite.*

5.2. **Inductive relations.** In Section 4.1 we have defined for fields R of banal characteristic with respect to G_n and G_m some explicit maps between supercuspidal supports:

$$\begin{array}{ccc} \Omega_R(G_m) & \to & \Omega_R(G_n) \\ (M,\rho)_{\mathrm{scs}} & \mapsto & (M \times T_{n-m}, \rho^\vee \chi_M \otimes_R \chi_{T_{n-m}})_{\mathrm{scs}} \end{array}$$

and where the characters are really explicit namely:

$$\chi_M = |\cdot|^{-\frac{n-m}{2}} \text{ and } \chi_{T_{n-m}} = |\cdot|_1^{(m+1-n)+\frac{(n-1)}{2}} \otimes_R \cdots \otimes_R |\cdot|_1^{\frac{(n-1)}{2}}.$$

Call this map $\theta_{n,m}(R) : \Omega_R(G_m) \to \Omega_R(G_n)$. It is a simple computation to check that these explicit maps give inductive relations such as:

$$\theta_{n,m}(R) = \theta_{n,k}(R) \circ \theta_{k,k}(R) \circ \theta_{k,m}(R) \text{ for } m \leqslant k \leqslant n.$$

We prove that these relations also holds in families over the Bernstein centres:

Proposition 5.4. *For $m \leqslant k \leqslant n$ we have inductive relations:*

$$\theta^{\#}_{n,m} = \theta^{\#}_{k,m} \circ \theta^{\#}_{k,k} \circ \theta^{\#}_{n,k}.$$

Proof. In order to prove this compatibility, we use induction in stages and the representation $W^m_{n,m}$ which is defining our morphisms. We can relate it to $W^k_{n,k}$ by an embedding of the trivial representation:

$$1_{n-m} \subseteq \mathrm{ind}_{P^{n-m}_{k-m}}^{G_{n-m}}(1_{k-m} \otimes 1_{n-k})$$

to obtain:

$$\begin{aligned} W^m_{n,m} = \mathrm{ind}_{P^n_m}^{G_n}(C_c^\infty(G_m) \otimes 1_{n-m}) &\subseteq \mathrm{ind}_{P^n_m}^{G_n}(C_c^\infty(G_m) \otimes \mathrm{ind}_{P^{n-m}_{k-m}}^{G_{n-m}}(1_{k-m} \otimes 1_{n-k})) \\ &= \mathrm{ind}_{P^n_{(m,k-m)}}^{G_n}(C_c^\infty(G_m) \otimes 1_{k-m} \otimes 1_{n-k}) \\ &= \mathrm{ind}_{P^n_k}^{G_n}(\mathrm{ind}_{P^k_m}^{G_k}(C_c^\infty(G_m) \otimes 1_{k-m}) \otimes 1_{n-k}) \\ &= \mathrm{ind}_{P^n_k}^{G_n}(W^m_{k,m} \otimes 1_{n-k}) \end{aligned}$$

Let $z \in \mathcal{Z}_{\mathbb{Z}[1/p]}(G_n)$. This element acts on $W^m_{n,m}$ as $z_{W^m_{n,m}} = \rho^m_r(\theta^{\#}_{n,m}(z)) = \rho^m_l((\theta^{\#}_{n,m}(z))^\vee)$ where ρ^m_l and ρ^m_r are the left and right action on $C_c^\infty(G_m)$. On the other hand, it acts on $\mathrm{ind}_{P^n_k}^{G_n}(W^m_{k,m} \otimes 1_{n-k})$ as $HC(z)_{W^m_{n,m} \otimes 1_{n-k}}$. In the course of defining $\theta^{\#}_{n,k} = \gamma_{\sigma_k} \circ HC$, the map γ_{σ_k} can alternatively be defined using $\eta_{1_{n-k}} : z \in \mathcal{Z}_{\mathbb{Z}[1/p]}(G_{n-k}) \mapsto z_{1_{n-k}} \in \mathbb{Z}[1/p]$ induced by the trivial character and $\mathrm{id} \otimes \eta_{1_{n-k}} : \mathcal{Z}_{\mathbb{Z}[1/p]}(G_k) \otimes \mathcal{Z}_{\mathbb{Z}[1/p]}(G_{n-k}) \to \mathcal{Z}_{\mathbb{Z}[1/p]}(G_k)$. Because:

$$z_{W^k_{n,k}} = \rho^k_l(\theta^{\#}_{n,k}(z)^\vee)$$

where $HC(z)_{\sigma_k} = \rho^k_l(\theta^{\#}_{n,k}(z)^\vee)$ we have $(\mathrm{id} \otimes \eta_{1_{n-k}})(HC(z)) = \theta^{\#}_{n,k}(z)^\vee$. So:

$$HC(z)_{W^m_{k,m} \otimes 1_{n-k}} = \big((\mathrm{id} \otimes \eta_{1_{n-k}})(HC(z))\big)_{W_{k,m}} = (\theta^{\#}_{n,k}(z)^\vee)_{W_{k,m}}$$

and by definition the latter is $\rho^m_r(\theta^{\#}_{k,m}(\theta^{\#}_{n,k}(z)^\vee))$. Using $\theta_{k,k} = (-)^\vee$ we obtained:

$$z_{\mathrm{ind}_{P^n_k}^{G_n}(W^m_{k,m} \otimes 1n-k)} = \rho^m_r(\theta^{\#}_{k,m} \circ \theta^{\#}_{k,m} \circ \theta^{\#}_{k,m}(z)).$$

By restriciton to the representation $W^m_{n,m} \subseteq \mathrm{ind}_{P^n_k}^{G_n}(W^m_{k,m} \otimes 1n-k)$ and by faithfulness of the action of $\mathcal{Z}_{\mathbb{Z}[1/p]}(G)$ on $W^m_{n,m}$, we deduce our inductive relations because $\rho^m_r(\theta^{\#}_{k,m} \circ \theta^{\#}_{k,k} \circ \theta^{\#}_{n,k}(z))$ agrees on $W^m_{n,m}$ with $\rho^m_r(\theta^{\#}_{n,m}(z))$. □

Remark 5.5. The proof of the latter proposition works over any $\mathbb{Z}[1/p]$-algebra and is explicit using subquotients of the Weil representation. There is an alternative shorter proof over $\mathbb{Z}[1/p]$ using the inductive relations $\theta_{n,k}(\mathbb{C}) \circ \theta_{k,k}(\mathbb{C}) \circ \theta_{k,m}(\mathbb{C}) = \theta_{n,m}(\mathbb{C})$ and a base change argument to $\mathbb{C}$ together with the compatibility of our maps to scalar extension.

Appendix A. Geometric lemma

Let G be a connected reductive group over F. We quickly explain why the geometric lemma still holds in the context of smooth representations with coefficients in R. Suppose we have fixed a minimal parabolic of G, say P_0, with Levi decomposition $P_0 = M_0N_0$. A parabolic subgroup P of G is said to be standard if it contains P_0. All such parabolics subgroups P come along with a standard Levi decomposition MN where M is the unique Levi in P containing M_0. Let $P' = M'N'$ be an other standard parabolic. For $(\sigma, V) \in \mathrm{Rep}_R(M)$, we are going to give a filtration of the restriction-induction $\mathfrak{r}_G^{M'} \circ \mathfrak{i}_M^G(\sigma) \in \mathrm{Rep}_R(M')$. This filtration is notoriously known as the geometric lemma. In order to define it, we need to introduce the following subset of the Weyl group W of G:

$$W^{M,M'} = \{w \in W \mid w(M \cap P_0) \subseteq P_0 \text{ and } w^{-1}(M' \cap P_0) \subseteq P_0\}.$$

By [Vig96, II.1.2], this set $W^{M,M'}$ also is a set of representatives for the double cosets $W_{M'}\backslash W/W_M$.

A.1. Non-normalised geometric lemma.

As we are not using normalised parabolic induction, because we are not assuming the existence of a square root of q in R, we recall the version of the geometric lemma we use:

Proposition A.1. *There exists a filtration of* $\mathfrak{r}_G^{M'} \circ \mathfrak{i}_P^G(\sigma) \in \mathrm{Rep}_R(M')$ *whose subquotients* $(I_w)_w$ *are indexed by* $W^{M,M'}$ *and given by:*

$$I_w \simeq \mathfrak{i}_{M' \cap w(M)}^{M'} \left(\delta_w \otimes_R (w \circ \mathfrak{r}_M^{w^{-1}(M') \cap M}(\sigma)) \right)$$

where $\delta_w = \delta_{N'}/\delta_{N' \cap w(P)}$ *is a character of* $M' \cap w(M)$.

We will not prove this proposition, but we refer to the many references [BZ77, Vig96, Ren09] for expositions on the geometric lemma. However, the most suitable reference to deal without normalisation seems to be the notes [Cas95, Sec 6]. We simply point out the precise results we need and their proofs go along the same way as in the notes. Let Ω_w be the double coset in $P\backslash G/P'$ associated to $w \in W^{M,M'}$. Choose a total order $<$ on $P\backslash G/P'$, or equivalently on $W^{M,M'}$, such that $U_w = \cup_{w' < w} \Omega_{w'}$ is an open subset of G for all $w \in W^M$. Denote the submodule of functions supported on U_w by $\mathfrak{i}_{U_w} = \{f \in \mathfrak{i}_M^G(\sigma) \mid \mathrm{supp}(f) \subseteq U_w\}$ and define $\mathfrak{j}_w = \mathfrak{i}_{U_w \cup \Omega_w}/\mathfrak{i}_{U_w}$. Then as in [Cas95, Prop 6.3.2], we have got in $\mathrm{Rep}_R(P)$:

$$\mathfrak{j}_w \simeq \mathrm{ind}_{P' \cap w(P)}^{P'}(w \circ \sigma)$$

and the computation of its N-coinvariants [Cas95, Props 6.2.1 & 6.3.3] is still valid so $(J_w)_N$ is the representation I_w we gave above.

A.2. Maximal parabolics for general linear groups.

The general linear group $G_n = \mathrm{GL}_n(F)$ is a connected reductive group over F. We choose as a minimal parabolic subgroup of G_n, also called a Borel subgroup in this situation, the subgroup of upper triangular matrices B_n with Levi decomposition T_nN_n where T_n is the subgroup of diagonal matices in G_n and N_n the set of unipotent matrices in B_k. For $0 \leqslant k \leqslant n$, set:

$$M_k^n = \left\{ \left[\begin{array}{cc} a_k & 0 \\ 0 & b_{n-k} \end{array} \right] \in G_n \;\middle|\; a_k \in G_k \text{ and } b_{n-k} \in G_{n-k} \right\}.$$

It is a standard Levi of G_k and it is contained in a unique standard parabolic subgroup denoted by $P_k^n = M_k^n N_k^n$. For $k \leqslant k' \leqslant n$, similarly write $P_{k'}^n = M_{k'}^n N_{k'}^n$.

Identify W and the permutation matrices representing $\mathfrak{S}_n$. By setting $r = \max(0, k-(n-k'))$, the map below induces an isomorphism between $W_{M_k^n}\backslash W/W_{M_{k'}^n} \simeq [\![r, k]\!]$:

$$\begin{array}{rcc} W & \to & [\![r, k]\!] \\ \sigma & \mapsto & |\{\sigma(1), \ldots, \sigma(k)\} \cap \{1, \ldots, k'\}| \end{array}.$$

As a result of this isomorphism, the set of representatives $W^{M_k^n, M_{k'}^n}$ is in bijection with the set $W(k,k',n) = \{w_{k,k',i}^n \mid 0 \leqslant i \leqslant \min(k, n-k')\}$ where:

$$w_{k,k',i}^n = \begin{bmatrix} \mathrm{Id}_{k-i} & & \\ & w_i & \\ & & \mathrm{Id}_{n-k'-i} \end{bmatrix} \in G_n \text{ with } w_i = \begin{bmatrix} 0 & \mathrm{Id}_i \\ \mathrm{Id}_{k'-k+i} & 0 \end{bmatrix} \in G_{k'-k+2i}.$$

These elements all satisfy $w_{k,k',i}^n(M_k^n \cap B_n) \subseteq B_n$ and $(w_{k,k',i}^n)^{-1}(M_{k'}^n \cap B_n) \subseteq B_n$. One has:

$$M_{k'}^n \cap w_{k,k',i}^n(M_k^n) = \begin{bmatrix} G_{k-i} & & & \\ & G_{k'-k+i} & & \\ & & G_i & \\ & & & G_{n-k'-i} \end{bmatrix}$$

that we denote $M_{(k-i,k'-k+i,i)}^n$ and:

$$M_k^n \cap (w_{k,k',i}^n)^{-1}(M_k^n) = \begin{bmatrix} G_{k-i} & & & \\ & G_i & & \\ & & G_{k'-k+i} & \\ & & & G_{n-k-i} \end{bmatrix}$$

denoted by $M_{(k-i,i,k'-k+i)}^n$.

Remark A.2. When $k' = k$ above, we write $w_{k,i}^n$ and $W(k,n)$ for short. The situation becomes simpler as the element $w_{k,i}^n$ has order at most 2 and is equal to its inverse.

A.3. **Comparing H-induced endomorphisms and G-endomorphisms.** Let G be a locally profinite group. Let H be a closed subgroup of G. In particular H is a locally profinite group as well. Let $V \in \mathrm{Rep}_R(H)$. For $f \in \mathrm{ind}_H^G(V)$ and $\varphi \in \mathrm{End}_H(V)$, define $\mathrm{ind}_H^G(\varphi) \cdot f \in \mathrm{ind}_H^G(V)$ by $(\mathrm{ind}_H^G(\varphi) \cdot f)(g) = \varphi(f(g))$ for all $g \in G$. Then it easy to see that:

Lemma A.3. *The map $\varphi \in \mathrm{End}_H(V) \mapsto \mathrm{ind}_H^G(\varphi) \in \mathrm{End}_G(\mathrm{ind}_H^G(V))$ is an injective morphism of algebras and the evaluation map $ev_1 : f \in \mathrm{ind}_H^G(V) \mapsto f(1_G) \in V$ induces a commutative diagramme:*

$$\begin{array}{ccc} \mathrm{ind}_H^G(V) & \xrightarrow{\mathrm{ind}_H^G(\varphi)} & \mathrm{ind}_H^G(V) \\ \downarrow ev_1 & & \downarrow ev_1 \\ V & \xrightarrow{\varphi} & V \end{array} .$$

We are specifically interested in situations when the previous injective map becomes an isomorphism, giving a canonical identification between $\mathrm{End}_H(V)$ and $\mathrm{End}_G(\mathrm{ind}_H^G(V))$.

Corollary A.4. *Suppose that $\mathrm{Hom}_H(\ker(ev_1)), V) = 0$. Then the map $\varphi \mapsto \mathrm{ind}_H^G(\varphi)$ above is an isomorphism and has inverse $\Phi \mapsto \tilde{\Phi}$ where $\tilde{\Phi}$ is, for $\Phi \in \mathrm{End}_G(\mathrm{ind}_H^G(V))$, the unique element in $\mathrm{End}_H(V)$ such that the following diagramme commutes:*

$$\begin{array}{ccc} \mathrm{ind}_H^G(V) & \xrightarrow{\Phi} & \mathrm{ind}_H^G(V) \\ \downarrow ev_1 & & \downarrow ev_1 \\ V & \xrightarrow{\tilde{\Phi}} & V \end{array} .$$

In particular $\Phi = \mathrm{ind}_H^G(\tilde{\Phi})$.

Proof. First of all we have that $\mathrm{End}_G(\mathrm{ind}_H^G(V)) \subseteq \mathrm{Hom}_G(\mathrm{ind}_H^G(V), \mathrm{Ind}_H^G(V))$ as the inclusion of induced representations $\mathrm{ind}_H^G(V) \subseteq \mathrm{Ind}_H^G(V)$ holds. Using Frobenius reciprocity, we get:

$$\mathrm{Hom}_G(\mathrm{ind}_H^G(V), \mathrm{Ind}_H^G(V)) \simeq \mathrm{Hom}_H(\mathrm{ind}_H^G(V), V).$$

The exact sequence $0 \to \ker(ev_1) \to \mathrm{ind}_H^G(V) \to V \to 0$ gives by right exactness of $\mathrm{Hom}_H(-, V)$:

$$\mathrm{Hom}_H(\ker(ev_1), V) \to \mathrm{Hom}_H(\mathrm{ind}_H^G(V), V) \xrightarrow{p} \mathrm{End}_H(V) \to 0.$$

As $\mathrm{Hom}_H(\ker(ev_1), V) = 0$, the map p must be an isomorphism. So on the one hand we have that $\varphi \in \mathrm{End}_H(V) \mapsto \varphi \circ ev_1 \in \mathrm{Hom}_H(\mathrm{ind}_H^G(V), V)$ is an isomorphism. On the other hand, the isomorphism coming from adjunction is $\psi \in \mathrm{Hom}_H(\mathrm{ind}_H^G(V), V) \mapsto A_\psi \in \mathrm{Hom}_G(\mathrm{ind}_H^G(V), \mathrm{Ind}_H^G(V))$ with $A_\psi(f) : g \mapsto \psi(g \cdot f))$ for $f \in \mathrm{ind}_H^G(V)$. Gathering together the previous two isomorphims yields an isomorphism:

$$\varphi \in \mathrm{End}_H(V) \mapsto A_{\varphi \circ ev_1} \in \mathrm{Hom}_H(\mathrm{ind}_H^G(V), \mathrm{Ind}_H^G(V)).$$

But the image of $A_{\varphi\circ ev_1}$ is included in $\mathrm{ind}_H^G(V)$. Indeed, we have $ev_1(g \cdot f) = f(g)$ for $f \in \mathrm{ind}_H^G(V)$ and $g \in G$, so $A_{\varphi\circ ev_1}(f) : g \mapsto \varphi(f(g))$ *i.e.* $A_{\varphi\circ ev_1} = i \circ \mathrm{ind}_H^G(\varphi)$ if i denotes $\mathrm{ind}_H^G(V) \subseteq \mathrm{Ind}_H^G(V)$. As a result $\varphi \in \mathrm{End}_H(V) \mapsto \mathrm{ind}_H^G(\varphi) \in \mathrm{End}_G(\mathrm{ind}_H^G(V))$ is an isomorphism of R-algebras. □

When the condition $\mathrm{Hom}_H(\ker(ev_1), V) = 0$ holds, we have got in particular a canonical isomorphism between $\mathrm{End}_H(V)$ and $\mathrm{End}_G(\mathrm{ind}_H^G(V))$. To refer to this very previse situation, we decorate isomorphims with curved arrows $\curvearrowright$ or $\curvearrowleft$ from $\mathrm{End}_H(V)$ to $\mathrm{End}_G(\mathrm{ind}_H^G(V))$. This means that $\mathrm{End}_H(V)$ acts on the set of "images" – some would prefer to say the "fiber" – of the representation $\mathrm{ind}_H^G(V)$ seen as a space of functions.

Appendix B. Around the Bernstein center

B.1. **Jacobson rings.** We are interested here in Jacobson (commutative) rings. By definition, they are the rings such that every prime ideal is the intersection of maximal ideals. In particular, their Jacobson radical – which is the intersection of all maximal ideals – agrees with their nilradical – which is the set of nilpotent elements, or equivalently, the intersection of all prime ideals. Any finitely generated (commutative) algebra over a Jacobson ring is itself Jacobson. A field is Jacobson, and so is the integers $\mathbb{Z}$, but $\mathbb{Z}_\ell$ is not as its Jacobson radical is $\ell\mathbb{Z}_\ell$.

When A is a Jacobson ring, the topological space $\mathrm{Spec}(A)$ is Jacobson – this is even an equivalence [Lem 10.35.4. in STACK EXCHANGE]. It ensures that closed points are somehow well-behaved with respect to subsets. For instance, if X is a locally closed subset of $\mathrm{Spec}(A)$, a closed point x in X will be closed in $\mathrm{Spec}(A)$ [Lem 5.18.5. in STACK EXCHANGE]. Denoting by $X_{\max}$ the set of closed points, we will have a natural identification $X_{\max} = X \cap \mathrm{Spec}(A)_{\max}$ and by the proof of [Lem 5.18.5. in STACK EXCHANGE] we have $X_{\max} \neq \emptyset$ if X is non-empty.

Lemma B.1. *Let A be a Jacobson reduced ring. Let U be an open dense subset of* $\mathrm{Spec}(A)$*. An element of A is determined by its specialisations over $U_{\max}$ i.e. we have an injective map:*

$$\begin{array}{ccc} A & \to & \prod_{\mathfrak{m}\in U_{\max}} A/\mathfrak{m} \\ a & \mapsto & (a_\mathfrak{m})_\mathfrak{m} \end{array}.$$

Proof. Consider the ideal $I = \cap_{\mathfrak{m}\in U_{\max}} \mathfrak{m}$, which is well-defined as $U_{\max} \neq \emptyset$. We want to show this ideal is the zero ideal. Let $f \in I$. By definition $D(f) = \mathrm{Spec}(A[1/f])$ is an open subset of $\mathrm{Spec}(A)$ and we have $D(f)_{\max} \cap U_{\max} = \emptyset$ by [Lemma 10.35.14. in STACK EXCHANGE]. This implies that $(D(f) \cap U)_{\max} = \emptyset$ and therefore $D(f) = \emptyset$ by density of U *i.e.* f is nilpotent in A by [GW20, Ex 2.2]. So $f = 0$ because A is reduced and we obtain $I = 0$ as claimed. □

For all connected reductive groups G over a local non-archimedean field F, the block decomposition of the center reads:

$$Z_\mathbb{C}(G) = \prod_{\mathfrak{s}\in\mathcal{B}_\mathbb{C}(G)} Z_\mathbb{C}^\mathfrak{s}(G)$$

where $\mathcal{B}_\mathbb{C}(G)$ is the set of inertial classes and each local component $Z_\mathbb{C}^\mathfrak{s}(G)$ is an integral domain that is finitely generated as a $\mathbb{C}$-algebra. In particular they are reduced Jacobson rings.

According to Lemma B.1 above, if M is a module over a reduced Jacobson ring A, it can be sufficient – when M is "big" enough – to check the action of A on any open dense subset to understand its action on M. We make this condition on M more precis by defining a quotient support $\mathrm{QS}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M \otimes_A A/\mathfrak{p} \neq 0\}$ for the module M. Note that, by Nakayama's lemma, this agrees with the usual definition of support when M is finitely generated. When $\mathfrak{m} \in \mathrm{Spec}(A)_{\max}$, an element $a \in A$ acts on $M \otimes_A A/\mathfrak{m}$ through a scalar $a_\mathfrak{m}(M) \in A/\mathfrak{m}$ in the center of $\mathrm{End}_{A/\mathfrak{m}}(M \otimes_A A/\mathfrak{m})$. Because $M \otimes_A A/\mathfrak{m}$ can be the zero module, we may have $a_\mathfrak{m}(M) = 0$ with this definition even though $a_\mathfrak{m} \neq 0$. The quotient support $\mathrm{QS}(M)$ is open in $\mathrm{Spec}(A)$ as its complement is easily seen to be closed.

Corollary B.2. *Let M be a module over a reduced Jacobson ring A. Assume that $\mathrm{QS}(M)$ is dense in $\mathrm{Spec}(A)$. Then for all open dense subset U of $\mathrm{Spec}(A)$, we have an injection:*

$$\begin{array}{ccc} A & \to & \prod_{\mathfrak{m} \in U_{\max}} A/\mathfrak{m} \\ a & \mapsto & (a_\mathfrak{m}(M))_\mathfrak{m} \end{array}.$$

Proof. This is a simple application of Lemma B.1 to the open dense set $U \cap \mathrm{QS}(M)$. ☐

B.2. **Generic semi-simplicity.** Let G be a connected reductive group over F. Let R be an algebraically closed field of characteristic ℓ that is banal with respect to G *i.e.* ℓ does not divide the pro-order $|G|$ of the group.

Lemma B.3. *The center of $\mathrm{Rep}_R(G)$ can be decomposed as a product over inertial classes:*

$$\mathfrak{z}_R(G) = \prod_{\mathfrak{s} \in \mathcal{B}_R(G)} \mathfrak{z}_R^\mathfrak{s}(G)$$

where each $\mathfrak{z}_R^\mathfrak{s}(G)$ is an integral domain and finite type R-algebra.

Proof. Bernstein Deligne. ☐

Proposition B.4. *There exists an open dense subset U of $\mathfrak{z}_R(G)$ such that for all $\eta \in U_{\max}$ the category $\mathrm{Rep}_R^\eta(G)$ is semi-simple and has a unique simple object π_η.*

Proof. Considering the block decomposition of $\mathrm{Rep}_R(G)$, it is enough to prove it for each block. So let $\mathfrak{s} \in \mathcal{B}_R(G)$. Let $P = MN$ and $\sigma_\mathfrak{s} \in \mathrm{Rep}_R(M)$ be a parabolic and an irreducible supercuspidal associated to this inertial class. The representation $i_P^G(\sigma_\mathfrak{s}\Psi)$ is a pro-generator of the category $\mathrm{Rep}_R^\mathfrak{s}(G)$ where $\Psi : M \to R[M/M^0]$ is the universal unramified character for M. Similarly to BERNSTEIN DELIGNE, there exists a compact open subgroup K in G of invertible pro-order in R and a non-zero $f \in \mathfrak{Z}_R^\mathfrak{s}(G)$ such that the R-algebra $\mathcal{H}_R(G,K)[1/f]$ is an Azumaya algebra over $\mathfrak{z}_R(G)[1/f]$ of dimension N. Here $\mathfrak{Z}_R(G)[1/f] = \mathfrak{Z}_R^\mathfrak{s}(G)[1/f]$ because $f \in \mathfrak{Z}_R^\mathfrak{s}(G)$. Furthermore $\mathrm{Rep}_R^\mathfrak{s}(G)$ is naturally equivalent to the category of modules over $\mathcal{H}_R^\mathfrak{s}(G,K)$ where $\mathcal{H}_R^\mathfrak{s}(G,K)$ is a direct factor ring of $\mathcal{H}_R(G,K)$. So $\mathcal{H}_R(G,K)[1/f] = \mathcal{H}_R^\mathfrak{s}(G,K)[1/f]$. Now specializing this algebra to a character $\eta : \mathcal{Z}_R^\mathfrak{s}(G) \to R$ gives an equivalence of categories between $\mathrm{Rep}_R^\eta(G)$ and the category of modules over $\mathcal{H}_R(G,K)[1/f] \otimes_\eta R \simeq \mathcal{M}_N(R)$. The category $\mathcal{M}_N(R)$-mod is Morita equivalent to the category of R-vector spaces. So we obtained that $D(f) = \mathrm{Spec}(\mathcal{Z}_R^\mathfrak{s}(G)[1/f])$ is a non-empty open set in the irreducible variety $\mathrm{Spec}(\mathcal{Z}_R^\mathfrak{s}(G))$, therefore it is dense and for all $\eta \in U_{\max}$ the category $\mathrm{Rep}_R^\eta(G)$ is semi-simple with a single simple object $\pi_\eta = i_P^G(\sigma\Psi) \otimes_\eta R$ coming from the generic irreducibility. ☐

B.3. **Regular representation.** We combine the previous two paragraphs to obtain "generic" properties about the regular representation. We carry on with the hypotheses with G connected reductive group over F and R an algebraically closed field of banal characteristic with respect to G. For $V \in \mathrm{Rep}_R(G)$ and $\eta : \mathfrak{z}_R(G) \to R$ a character of the center, we recall that the biggest η-quotient of V is defined as $V_\eta = V \otimes_\eta R = V/V[\eta]$.

Proposition B.5. *There exists an open dense subset U in $\mathrm{Spec}(Z_R(G))$ such that for all characters $\eta : \mathfrak{z}_R(G) \to R$ in $U_{\max}$, we have:*

$$C_c^\infty(G)_\eta = \pi_\eta \otimes_R \pi_\eta^\vee.$$

Proof. This is an easy application of Proposition B.4 combined with the classical fact that $C_c^\infty(G)_\pi = \pi \otimes_R \pi^\vee$ as a $(G \times G)$-representation for all irreducible $\pi \in \mathrm{Rep}_R(G)$. ☐

B.4. **Extension of scalars.** Let A be a $\mathbb{Z}[1/p]$-algebra. We first introduce the Gelfand-Graev representations that will be our cornerstone for the compatibility of the Bernstein centre with scalar extension. In this section, all tensor products are over $\mathbb{Z}[1/p]$ unless otherwise stated.

Let N_n be the unipotent radical of the standard Borel *i.e.* the group of unipotent upper triangular matrices in G_n. We consider the ring $R_0 = \mathbb{Z}[1/p, \mu_{p^\infty}]$ that is obtained by adjoining all p-power roots of unity. Let ψ be a non-degenerate character of N with values in R_0. We define the Gelfand-Graev representation with coefficients in R_0 by $\mathrm{ind}_{N_n}^{G_n}(\psi)$. We introduce the term *locally finitely generated* for a representation $V \in \mathrm{Rep}_A(G_n)$. It means, in terms of the level decomposition $V = \bigoplus_r V_r$, that each $V_r \in \mathrm{Rep}_A^r(G_n)$ is finitely generated. By a *local progenerator* we therefore mean a locally finitely generated projective generator of $\mathrm{Rep}_A(G_n)$. The forthcoming paper [DHKM23] proves that:

Proposition B.6. *There exists an integral model $W_{n,\psi}$ of the representation of Gelfand-Graev over $\mathbb{Z}[1/p]$ such that $W_{n,\psi} \otimes R_0$ is isomorphic to $\mathrm{ind}_{N_n}^{G_n}(\psi)$. Furthermore $W_{n,\psi}$ is locally finitely generated and projective.*

Definition B.7. *Over a $\mathbb{Z}[1/p]$-algebra A, the Gelfand-Graev representation is $W_{n,\psi}^A = W_{n,\psi} \otimes A$.*

We are going to prove:

Theorem B.8. *The natural map $\Phi_A : z \in Z_A(G_n) \mapsto z_{\mathrm{W}_{N,\psi}^A} \in \mathrm{End}_G(W_{N,\psi}^A)$ is an isomorphism.*

The proof of the theorem breaks down into the following two lemmas, which easily implies on the one hand the surjectivty of Φ_A and on the other hand its injectivity. As these proofs require consequent developments, we prove them in a separate section:

Lemma B.9. *There exists a section $\Psi_A : \mathrm{End}_G(W_{N,\psi}^A) \to Z_A(G_n)$ of Φ_A.*

Lemma B.10. *The natural action of $Z_A(G_n)$ on $W_{N,\psi}^A$ is faithful.*

We deduce the compatibility to scalar extensions as a corollary of Theorem B.8. Because finitely generated and projective imply finitely presented, the endomorphisms of the Gelfand-Graev representation is compatible with arbitrary scalar extensions according to [Lam06, Prop I.2.13]. This means that the functor $- \otimes A$ induces an isomorphism:

$$\mathrm{End}_{\mathbb{Z}[1/p][G]}(W_{N,\psi}) \otimes A \simeq \mathrm{End}_{A[G_n]}(W_{N,\psi}^A).$$

This gives us an isomorphism $Z_{\mathbb{Z}[1/p]}(G_n) \otimes A \simeq Z_A(G_n)$ thanks to Theorem B.8. However, there is a more intrinsic way to describe this isomorphism. Consider the natural map:

$$Z_{\mathbb{Z}[1/p]}(G_n) \to Z_A(G_n)$$

induced by the forgetful functor $F : \mathrm{Rep}_A(G_n) \to \mathrm{Rep}_{\mathbb{Z}[1/p]}(G_n)$. To describe it explicitly, this ring morphism is $z \mapsto (z_{F(V)})_V$ where V runs over all representations in $\mathrm{Rep}_A(G_n)$. The only thing one needs to check is the fact that for all $f \in \mathrm{Hom}_{A[G_n]}(V, V')$ we have $z_{F(V')} \circ f = f \circ z_{F(V)}$. But the forgetful functor is faithful and it holds applying it to f because $z \in Z_{\mathbb{Z}[1/p]}(G_n)$. This also proves that $z_{F(V)}$ is A-linear by taking $V = V'$ and f the multiplication by $a \in A$. Of course this natural map induces a bilinear map $Z_{\mathbb{Z}[1/p]}(G_n) \times A \to Z_A(G_n)$ which factors through the tensor product:

$$\eta_A : Z_{\mathbb{Z}[1/p]}(G_n) \otimes_{\mathbb{Z}[1/p]} A \to Z_A(G_n).$$

As a consequence of the present discussion, we have:

Corollary B.11. *The map η_A is an isomorphism.*

B.5. **Proofs of Lemmas B.9 and B.10.** Central to our approach is the construction of a local progenerator of $\mathrm{Rep}_A(G_n)$ out of the Gelfand-Graev representation. First of all, we state a general result about compatibility of progenerators with scalar extension and faithfully flat descent. In the lemma below G is a connected reductive group:

Lemma B.12. *Let $A \to B$ be a ring morphism and $P \in \mathrm{Rep}_A(G)$.*

(i) *If P is a local progenerator of $\mathrm{Rep}_A(G)$, then $P \otimes_A B$ is a local progenerator of $\mathrm{Rep}_B(G)$.*

(ii) *If* $P \otimes_A B$ *is a local progenerator of* $\mathrm{Rep}_B(G)$ *and* B *is faithfully flat over* A*, then* P *is a local progenerator of* $\mathrm{Rep}_A(G)$*.*

Proof. (i) If $P = \bigoplus P_r$ is the level decomposition of P, then the level decomposition of $P \otimes_A B$ in $\mathrm{Rep}_B(G)$ is given by $(P \otimes_A B)_r = P_r \otimes_A B$. So being locally finitely generated is preserved by scalar extension. Now by the tensor-hom adjunction, we have a canonical isomorphism between the functors $\mathrm{Hom}_{B[G]}(P \otimes_A B, -)$ and $\mathrm{Hom}_{A[G]}(P, \mathrm{Hom}_B(B, -))$, where the latter is exact as $\mathrm{Hom}_B(B, -)$ is the forgetful functor $\mathrm{Rep}_B(G) \to \mathrm{Rep}_A(G)$. The fact that $\mathrm{Hom}_{B[G]}(P \otimes_A B, -)$ is faithful can be easily seen using again the tensor-hom adjunction as it becomes the composition of two faithful functors. So $P \otimes_A B$ is a generator.

(ii) First we prove $(P \otimes B)_r$ is finitely presented as a $B[G]$-module. Suppose $y_1, \ldots, y_m$ are generators of $(P \otimes B)_r = P_r \otimes B$. If we write $y_i = \sum_j x_{i,j} \otimes b_{i,j}$ for $x_{i,j} \in P$ and $b_{i,j} \in B$, then the set $\{x_{i,j} \otimes 1\}_{i,j}$ is also a set of generators for $(P \otimes B)_r$. Rewriting the set $\{x_{i,j}\}_{i,j} = \{x_1, \ldots, x_n\}$, choose a compact open subgroup H fixing all the x_i's. Consider the exact sequence of smooth $B[G]$-modules

$$0 \to K \to B[G/H]^{\oplus n} \to (P \otimes B)_r \to 0$$

where K is the kernel of the map $(\beta_i)_i \mapsto \sum_i \beta_i(x_i \otimes 1)$. Since $(P \otimes B)_r$ is projective, this exact sequence splits and we can write $B[G/H]^{\oplus n} = K \oplus (P \otimes B)_r$. In particular, K is finitely generated because it is the quotient of a finitely generated object. We conclude $(P \otimes B)_r$ is finitely presented.

Next, we argue that P_r is finitely presented. Consider the map $A[G/H]^{\oplus n} \to P_r$ sending $(\alpha_i)_i$ to $\sum_i \alpha_i x_i$ and let K' be its kernel. Since it is a map of smooth $A[G]$ modules that is surjective after a faithfully flat extension, it is surjective, and P_r is finitely generated. Since faithfully flat extensions preserve kernels, we have $K' \otimes_A B = K$, and we can repeat the previous argument to conclude K' is also finitely generated. Thus P_r is finitely presented.

Suppose $V \to W$ is a surjective map in $(\mathrm{Rep}_A(G))_r$. Let C denote the cokernel of

$$\mathrm{Hom}_{A[G]}(P_r, V) \to \mathrm{Hom}_{A[G]}(P_r, W).$$

Since P_r is finitely presented and $A \to B$ is flat, we have $\mathrm{Hom}_{A[G]}(P_r, V) \otimes B \to \mathrm{Hom}_{B[G]}((P \otimes B)_r, V \otimes B)$ is an isomorphism [Lam06, Prop 2.13] and similarly for W. Thus $C \otimes B$ is the cokernel of the map $\mathrm{Hom}_{B[G]}((P \otimes B)_r, V \otimes B) \to \mathrm{Hom}_{B[G]}((P \otimes B)_r, W \otimes B)$, which is zero by projectivity of $(P \otimes B)_r$. Hence C is zero by faithful flatness of $A \to B$. We conclude that P_r is projective.

We now turn to showing that finitely presented and flat implies projective in the category of smooth $A[G]$-modules. Let H be a compact open subgroup fixing a finite set of generators of P_r and their relations. This amounts to choosing an exact sequence $0 \to K \to F \to P_r \to 0$ where $F = A[G/H]^{\oplus l}$ for some l and K is generated by a set $c_1, \ldots, c_n$ of elements that are fixed by H. By the equational criterion of flatness there exists a homomorphism $\theta : F \to K$ such that $\theta(c_i) = c_i$ for all i ([Lam99, Thm 4.23]. Thus θ is the identity on K and defines a section of $K \to F$, so the exact sequence splits. We conclude that P_r is a direct summand of F, which is projective, hence P_r is also projective.

For faithfulness we need only show that $\mathrm{Hom}_{A[G]}(P_r, \pi) \neq 0$ for all simple objects in the depth r subcategory. Since π is simple, its endomorphism ring has no zero divisors, so π is equivalent to a module with coefficients in a residue field $k(\mathcal{P})$ for $\mathcal{P} \in \mathrm{Spec}(A)$. By the tensor-hom adjunction we have $\mathrm{Hom}_{A[G]}(P_r, \pi) \cong \mathrm{Hom}_{k(\mathcal{P})[G]}(P_r \otimes_A k(\mathcal{P}), \pi)$ and we must check that $P_r \otimes_A k(\mathcal{P})$ is faithful in $\mathrm{Mod}_{k(\mathcal{P})}(G)$. By faithful flatness, $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, and taking $\mathcal{Q}$ lying over $\mathcal{P}$, we have

$$\mathrm{Hom}_{k(\mathcal{Q}[G]}((P \otimes B)_r \otimes_B k(\mathcal{Q}), \pi \otimes k(\mathcal{Q})) = \mathrm{Hom}_{k(\mathcal{P})[G]}(P \otimes_A k(\mathcal{P}), \pi) \otimes_{k(\mathcal{P})} k(\mathcal{Q}).$$

Since the left side is nonzero by faithfulness of $(P \otimes B)_r$, we conclude that

$$\mathrm{Hom}_{k(\mathcal{P})[G]}(P \otimes_A k(\mathcal{P}), \pi) \neq 0.$$

□

The first part of the lemma reduces the problem to constructing a progenerator with coefficients in $\mathbb{Z}[1/p]$. The second part tells us that it is enough to verify over a faithfully flat extension such as R_0. Here is the progenerator we mentioned for general linear groups:

Lemma B.13. *Let* $\underline{\mathrm{std}}$ *be the set of standard parabolic subgroups of* G *and define:*

$$W_{N,\psi}^{\mathrm{gen}} = \bigoplus_{P \in \underline{\mathrm{std}}} \mathrm{ind}_{\bar{P}}^{G} \circ \mathrm{res}_{G}^{\bar{P}}(W_{N,\psi}).$$

It is a local progenerator of $\mathrm{Rep}_{\mathbb{Z}[1/p]}(G)$ *and therefore we have:*

$$Z_{\mathbb{Z}[1/p]}(G) = Z(\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}})).$$

Proof. According to Lemma B.12, it is enough to prove it after a faithfully flat base change as induction and restriction functors commute to scalar extension. Therefore it is enough to prove it over R_0 replacing $W_{N,\psi}$ by $\mathrm{ind}_N^G(\psi)$. The representation $\mathrm{ind}_N^G(\psi)$ is locally finitely generated and projective by Proposition B.6. Note that the induction (and restriction) functors preserve finitely generated represenation over arbitrary ring, as opposed to what is written in [DHKM22, Cor 1.5]: indeed the noetherianity hypothesis there is superfluous as it relies on DAT and second adjunction[1], which are both valid over any $\mathbb{Z}[1/p]$-algebra. So $\mathrm{ind}_N^G(\psi)^{\mathrm{gen}}$ is locally finitely generated and projective. We now prove it is a generator. As all finitely generated objects admits a simple quotient, it is enough to prove that for all simple objects $\pi \in \mathrm{Rep}_{R_0}(G)$:

$$\mathrm{Hom}_{R_0[G]}(\mathrm{ind}_N^G(\psi)^{\mathrm{gen}}, \pi) \neq 0.$$

Actually π has coefficients in a residue field of R_0. Denoting by $\mathcal{P} \in \mathrm{Spec}(R_0)$ the prime ideal kernel of $R_0 \to \mathrm{End}_{R_0[G]}(\pi)$ given by the action of scalars, we have $k(\mathcal{P}) = \mathrm{Frac}(R_0/\mathcal{P})$ and $\pi \in \mathrm{Rep}_{k(\mathcal{P})}(G)$. Therefore by tensor-hom adjunction:

$$\mathrm{Hom}_{R_0[G]}(\mathrm{ind}_N^G(\psi)^{\mathrm{gen}}, \pi) \simeq \mathrm{Hom}_{k(\mathcal{P})[G]}(\mathrm{ind}_N^G(\psi)^{\mathrm{gen}} \otimes_{R_0} k(\mathcal{P}), \pi).$$

So we reduce the question to checking that $\mathrm{ind}_N^G(\psi)^{\mathrm{gen}} \otimes_{R_0} k(\mathcal{P}) = \mathrm{ind}_N^G(\psi_{k(\mathcal{P})})^{\mathrm{gen}}$ is a local progenerator of the category $\mathrm{Rep}_{k(\mathcal{P})}(G)$. Note that field extensions are faithfully flat, so we can always assume that our base field k is algebraically closed. Now we obtain that $\mathrm{ind}_N^G(\psi_k)^{\mathrm{gen}}$ is a local progenerator as a consequence of the following three properties:

- the existence of cuspidal support;
- all cuspidals are generic for general linear groups;
- the restriction $r_G^{\bar{P}}(\mathrm{ind}_N^G(\psi_k)) \simeq \mathrm{ind}_{N_M}^M(\psi_k|_{N_M})$ where P has Levi M and $N_M = N \cap M$.

So $\mathrm{Hom}_{R_0[G]}(\mathrm{ind}_N^G(\psi)^{\mathrm{gen}}, \pi) \neq 0$ and by faithfully flat descent $W_{N,\psi}^{\mathrm{gen}}$ is a local progenerator. □

Let $E_P \in \mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}})$ be the projection on the direct factor associated to $P \in \underline{\mathrm{std}}$. It is an idempotent and the commutor of all these idempotents is:

$$\bigoplus_{P \in \underline{\mathrm{std}}} \mathrm{End}_G(\mathrm{ind}_{\bar{P}}^{G} \circ \mathrm{res}_{G}^{\bar{P}}(W_{N,\psi})).$$

Therefore we have clear inclusions:

$$Z(\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}})) \subseteq \bigoplus_{P \in \underline{\mathrm{std}}} \mathrm{End}_G(\mathrm{ind}_{\bar{P}}^{G} \circ \mathrm{res}_{G}^{\bar{P}}(W_{N,\psi})) \subseteq \mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}}).$$

We denote by $(M_P)_{P \in \underline{\mathrm{std}}}$ an element in the second endomorphism ring and consider the map $\mathrm{ev}_G : (M_P)_{P \in \underline{\mathrm{std}}} \mapsto M_G$, as well as the restriction Φ of ev_G to $Z(\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}}))$. The functor $F = \bigoplus_{P \in \underline{\mathrm{std}}} \mathrm{ind}_{\bar{P}}^{G} \circ \mathrm{res}_{G}^{\bar{P}}$ induces a morphism of A-algebras:

$$\Psi_F : \phi \mapsto (\phi_P)_{P \in \underline{\mathrm{std}}} = (\mathrm{ind}_{\bar{P}}^{G} \circ \mathrm{res}_{G}^{\bar{P}}(\phi))_{P \in \underline{\mathrm{std}}}.$$

We want to show:

[1]one of the reasons that could explain it: the proof of the second adjunction in [DHKM22] initially dealt with noetherian $\mathbb{Z}[1/p]$-algebras and they relaxed the noetherianity assumption at a very late stage of the writing.

Lemma B.14. *The image of Ψ_F is central in $\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}})$ i.e. there exists a section Ψ of Φ completing the commutative diagram of solid arrows:*

$$\begin{array}{ccccc}
\mathrm{End}_G(W_{N,\psi}) & \overset{\Psi}{\dashrightarrow} & Z(\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}})) & \xrightarrow{\Phi} & \mathrm{End}_G(W_{N,\psi}) . \\
& \searrow^{\Psi_F} & \big\downarrow & \nearrow_{\mathrm{ev}_G} & \\
& & \bigoplus_{P\in\underline{\mathrm{std}}} \mathrm{End}_G(\mathrm{ind}_{\bar{P}}^{G}\circ \mathrm{res}_{G}^{\bar{P}}(W_{N,\psi})) & & \\
& & \big\downarrow & & \\
& & \mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}}) & &
\end{array}$$

Morevoer Φ is injective, therefore both Ψ and Φ are isomorphisms.

Proof. We start by the injectivity of Φ. As $W_{N,\psi}^{\mathrm{gen}}$ is a local progenerator, we know that the map $z \in Z_{\mathbb{Z}[1/p]}(G) \mapsto z_{W_{N,\psi}^{\mathrm{gen}}} \in Z(\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}}))$ is an isomorphism, and composing by Φ we obtain $z \in Z_{\mathbb{Z}[1/p]}(G) \mapsto z_{W_{N,\psi}} \in \mathrm{End}_G(W_{N,\psi})$. Similarly to HELM-WHITTAKER, extending the scalar to $\mathbb{C}$ gives the injectivity of Φ.

There remains to prove the existence of the section Ψ. The composition $\Psi_F \circ \Phi$ induces:

$$z \in Z_{\mathbb{Z}[1/p]}(G) \mapsto (\mathrm{ind}_{\bar{P}}^{G}\circ \mathrm{res}_{G}^{\bar{P}}(z_{W_{N,\psi}}))_{P\in\underline{\mathrm{std}}} \in \bigoplus_{P\in\underline{\mathrm{std}}} \mathrm{End}_G(\mathrm{ind}_{\bar{P}}^{G}\circ \mathrm{res}_{G}^{\bar{P}}(W_{N,\psi})).$$

We want to prove that the latter is central. First the action of $z \in Z_{\mathbb{Z}[1/p]}(G)$ on $W_{N,\psi}^{\mathrm{gen}}$ is given by $(z_{\mathrm{ind}_{\bar{P}}^{G}\circ\mathrm{res}_{G}^{\bar{P}}(W_{N,\psi})})_{P\in\underline{\mathrm{std}}} \in Z(\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}}))$. The centrality will be a clear consequence of the following identity:

$$z_{\mathrm{ind}_{\bar{P}}^{G}\circ\mathrm{res}_{G}^{\bar{P}}(W_{N,\psi})} = \mathrm{ind}_{\bar{P}}^{G}\circ \mathrm{res}_{G}^{\bar{P}}(z_{W_{N,\psi}}).$$

This identity comes from the existence of Harish-Chandra morphisms [DHKM22, Th 4.1]. Actually the only property we use, which is weaker than the full results of Harish-Chandra morphisms, is the fact that $z_{\mathrm{ind}_{\bar{P}}^{G}\circ\mathrm{res}_{G}^{\bar{P}}(W_{N,\psi})} = \mathrm{ind}_{\bar{P}}^{G}(f')$ for some $f' \in \mathrm{End}_M(\mathrm{res}_{G}^{\bar{P}}(W_{N,\psi}))$. By Frobenius reciprocity:

$$f' = \mathrm{res}_{G}^{\bar{P}}(z_{W_{N,\psi}})$$

and therefore the identity $z_{W_{N,\psi}^{\mathrm{gen}}} = \Psi_F \circ \Phi(z_{W_{N,\psi}^{\mathrm{gen}}})$ holds. So Ψ_F induces the required Ψ, which is at the same time injective and a section of Φ. Hence Ψ and Φ are isomorphisms. ☐

We can extend the scalar to A to obtain a new diagam. By taking into account compatibilies to scalar extension already mentioned [Lam06, Prop I.2.13], we obtain a commutative diagram:

$$\begin{array}{ccccc}
\mathrm{End}_G(W_{N,\psi}^{A}) & \xrightarrow{\Psi\otimes A} & Z(\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}}))\otimes A & \xrightarrow{\Phi\otimes A} & \mathrm{End}_G(W_{N,\psi}^{A}) \\
& \searrow^{\Psi_{F_A}} & \big\downarrow & \nearrow_{\mathrm{ev}_G} & \\
& & \bigoplus_{P\in\underline{\mathrm{std}}} \mathrm{End}_G(\mathrm{ind}_{\bar{P}}^{G}\circ \mathrm{res}_{G}^{\bar{P}}(W_{N,\psi}^{A})) & & \\
& & \big\downarrow & & \\
& & \mathrm{End}_G((W_{N,\psi}^{A})^{\mathrm{gen}}) & &
\end{array}$$

where the endofunctors $F_A = \bigoplus_{P\in\underline{\mathrm{std}}} \mathrm{ind}_{\bar{P}}^{G}\circ \mathrm{res}_{G}^{\bar{P}}$ and $F\otimes A$ of $\mathrm{Rep}_A(G_n)$ are canonically isomorphic to because restriction and induction functors commute with scalar extension. Also $\Psi\otimes A$ and $\Phi\otimes A$ are still inverse isomorphisms and Ψ_{F_A} remains a section of ev_G.

Because $\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}}) \otimes A$ identifies with $\mathrm{End}_G((W_{N,\psi}^A)^{\mathrm{gen}})$, the image of the first vertical map must lie in $Z(\mathrm{End}_G((W_{N,\psi}^A)^{\mathrm{gen}}))$. In other words, we can complete the diagram with a section Ψ_A and a retraction Φ_A, coming respectively from $\Psi \otimes A$ and ev_G, into the following:

$$\begin{array}{ccccc}
\mathrm{End}_G(W_{N,\psi}^A) & \xrightarrow{\Psi\otimes A} & Z(\mathrm{End}_G(W_{N,\psi}^{\mathrm{gen}})) \otimes A & \xrightarrow{\Phi\otimes A} & \mathrm{End}_G(W_{N,\psi}^A)\ . \\
 & \searrow^{\Psi_A} & \downarrow & \nearrow^{\Phi_A} & \\
 & \Psi_{F_A} & Z(\mathrm{End}_G(W_{N,\psi}^A)^{\mathrm{gen}})) & \mathrm{ev}_G & \\
 & \searrow & \downarrow & \nearrow & \\
 & & \bigoplus_{P\in\underline{\mathrm{std}}} \mathrm{End}_G(\mathrm{ind}_{\bar{P}}^G \circ \mathrm{res}_G^{\bar{P}}(W_{N,\psi}^A)) & & \\
 & & \downarrow & & \\
 & & \mathrm{End}_G((W_{N,\psi}^A)^{\mathrm{gen}}) & &
\end{array}$$

In particular composing from left to right implies Lemma B.9.

There remains to prove Lemma B.10 to have the compatibility to arbitrary scalar extension:

$$Z_{\mathbb{Z}[1/p]}(G) \otimes A \simeq Z_A(G).$$

Note that the centre $Z_A(G)$ acts faithfully on $W_{N,\psi}$ is equivalent to the injectivity of Φ_A. In the course of the proof of Lemma B.14, we proved the injectivity of Φ using an identity that was a consequence of the existence of Harish-Chandra morphisms:

$$z_{\mathrm{ind}_{\bar{P}}^G \circ \mathrm{res}_G^{\bar{P}}(W_{N,\psi})} = \mathrm{ind}_{\bar{P}}^G \circ \mathrm{res}_G^{\bar{P}}(z_{W_{N,\psi}}).$$

for $z \in Z_{\mathbb{Z}[1/p]}(G)$. If such an identity was holding for $z \in Z_A(G)$ and $W_{N,\psi}^A$, we would be able to conclude as earlier that $\Psi_A \circ \Phi_A(z_{(W_{N,\psi}^A)^{\mathrm{gen}}}) = z_{(W_{N,\psi}^A)^{\mathrm{gen}}}$. In this case Ψ_A is an isomorphism and so does Φ_A. Therefore we focus our efforts on proving this identity:

Proposition B.15. *Let $z \in Z_A(G)$ and $P \in \underline{\mathrm{std}}$. Then:*

$$z_{\mathrm{ind}_{\bar{P}}^G \circ \mathrm{res}_G^{\bar{P}}(W_{N,\psi})} = \mathrm{ind}_{\bar{P}}^G \circ \mathrm{res}_G^{\bar{P}}(z_{W_{N,\psi}}).$$

Proof. As in the proof of Lemma B.14, it is sufficient to prove that:

$$z_{\mathrm{ind}_{\bar{P}}^G \circ \mathrm{res}_G^{\bar{P}}(W_{N,\psi})} = \mathrm{ind}_{\bar{P}}^G(f') \text{ for some } f' \in \mathrm{End}_M(\mathrm{res}_G^{\bar{P}}(W_{N,\psi}^A)).$$

The methods of Bushnell-Henniart [DHKM23] give that $\mathrm{res}_G^{\bar{P}}(W_{N,\psi}^A)$ is the Gelfand-Graev representation for the Levi M and we denote it by σ. We want to prove that:

$$Z(\mathrm{End}_G(\mathrm{ind}_{\bar{P}}^G(\sigma))) \subseteq \mathrm{ind}_{\bar{P}}^G(\mathrm{End}_M(\sigma)).$$

There is no chance for $\mathrm{End}_G(\mathrm{ind}_{\bar{P}}^G(\sigma))$ to satisfy this inclusion, even in the complex setting. Indeed when $A = \mathbb{C}$, we can consider $G = G_2$ and the principal block component of the Gefand-Graev representation is $\mathrm{ind}_{\bar{B}}^G(\mathbb{C}[T_2/T_2^0])$ which is G-isomorphic to $\mathrm{ind}_I^G(1_I)$ where I is the Iwahori group; here $\mathbb{C}[T_2/T_2^0]$ is the variety of unramified characters as well as the level 0 component of $\mathrm{res}_{G_2}^{\bar{B}_2}(W_{2,\psi})$ which is the Gelfand-Graev representation of T_2. So we have $\mathrm{End}_G(\mathrm{ind}_{\bar{B}}^G(\mathbb{C}[T_2/T_2^0]))$ is isomorphic to the Hecke-Iwahori algebra $\mathcal{H}(G,I)^{\mathrm{op}}$ by the isomorphism with $\mathrm{ind}_I^G(1_I)$. Nevertheless, the commutant of $E_{\mathrm{ind}} = \mathrm{ind}_{\bar{P}}^G(\mathrm{End}_{T_2}(\mathbb{C}[T_2/T_2^0]))$ in $E = \mathrm{End}_G(\mathrm{ind}_{\bar{B}}^G(\mathbb{C}[T_2/T_2^0]))$ is E_{ind} itself as a consequence of the generic irreducibility theorem.

We are looking for a different approach that does not rely on the generic irreducibility, but computes the commutant of E_{ind} in E using different arguments. To say a motivational statement, the generic irreducibility deals with generic points and is blind to spot action of nilpotent elements of the centre. Our main tool will be an explanation of the geometric lemma that keeps track of the action of the induced endomorphisms. We denote by $E_{\mathrm{ind}} = \mathrm{ind}_{\bar{P}}^G(\mathrm{End}_M(\sigma))$ the induced endomorphisms in the full endomorphism ring $E = \mathrm{End}_G(\mathrm{ind}_{\bar{P}}^G(\sigma))$. Our claim deals with the commutant $C_E(E_{\mathrm{ind}})$ of E_{ind} in E and is the following:

Lemma B.16. $E_{\mathrm{ind}} = C_E(E_{\mathrm{ind}})$.

Proof. We consider $\mathrm{ind}_{\bar{P}}^G(\sigma)$ endowed with its $E_{\mathrm{ind}} \times G$-structure:

$$(\mathrm{ind}(\varphi), g_0) \cdot f : g \in G \mapsto \varphi(f(gg_0)) \in \sigma.$$

We want to determine $C_E(E_{\mathrm{ind}}) = \mathrm{End}_{E_{\mathrm{ind}} \times G}(\mathrm{ind}_{\bar{P}}^G(\sigma))$. By Frobenius reciprocity, an element of this endomorphism ring is determined by its composition with ev_1. Therefore E_{ind} is naturally contained in $\mathrm{Hom}_{E_{\mathrm{ind}} \times M}(\mathrm{ind}_{\bar{P}}^G(\sigma)|_M, \sigma)$ that is isomorphic to $C_E(E_{\mathrm{ind}})$. Note that $z \in Z(M)$ determines a $\sigma(z) \in \mathrm{End}_M(\sigma)$. So on σ the $E_{\mathrm{ind}} \times G$-action must factor through $(\sigma(z), 1_G) - (\mathrm{Id}_\sigma, z)$ because $\mathrm{ev}_1((\sigma(z), 1_G) \cdot f) = \mathrm{ev}_1((\mathrm{Id}_\sigma, z) \cdot f) = f(z) = \sigma(z)(f(1))$. We can consider the subquotients of $\mathrm{ind}_{\bar{P}}(\sigma)|_M$ given by restriction to the subquotient I_w supported on a Bruhat cell $\bar{P}w\bar{P}$. The quotient ev_1 given by σ corresponds to the Bruhat cell $\bar{P}$. For other Bruhat cells, we have I_w is the set of smooth functions $f : \bar{P}w\bar{P} \to \sigma$ which are compactly supported modulo $\bar{P}$ (*i.e.* the image of $\mathrm{supp}(f)$ in $\bar{P}\backslash\bar{P}w\bar{P}$ is compact) and left equivariant in the usual sense $f(m\bar{n}w\bar{p}) = \sigma(m)f(w\bar{p})$. It is stable under the action of $E_{\mathrm{ind}} \times M$ inherited from $\mathrm{ind}_{\bar{P}}^G(\sigma)|_M$. In particular such an $f \in I_w$ is determined by its restriction to $w\bar{P}$. A simple computation yields for $z \in Z(M)$ that:

$$\Big((\mathrm{Id}_\sigma, z) \cdot f\Big)(w\bar{p}) = f(w\bar{p}z) = f(wzw^{-1}w\bar{p}) = \Big((\sigma(wzw^{-1}), 1_G) \cdot f\Big)(w, \bar{p}).$$

So $(\sigma(wzw^{-1}), 1_G) - (\mathrm{Id}_\sigma, z)$ acts trivially on I_w.

Now in order for this $E_{\mathrm{ind}} \times M$-action on I_w to be compatible with that on σ, any morphism in $\mathrm{Hom}_{E_{\mathrm{ind}} \times M}(I_w, \sigma)$ must factor through:

$$I'_w = I_w / \langle (\sigma(z) - \sigma(wzw^{-1}), 1_G) \cdot f \rangle = I_w \otimes_{E_{\mathrm{ind}}} (E_{\mathrm{ind}}/J)$$

where $J = \langle \sigma(z) - \sigma(wzw^{-1}) \mid z \in Z(M) \rangle$ is an ideal in E_{ind}. This ideal is really big in some sense: writing $\Psi_1 \otimes \Psi_2 = \mathbb{C}[T_2/T_2^0]$ in the example of the principal block, the ideal J will be the Zariski closed locus $\Psi_1 = \Psi_2$ whose complement is the dense open $\Psi_1 \neq \Psi_2$. In particular, it will introduce some strong condition (torsion or trivial action) on the action of E_{ind} on I'_w that won't be satisfied for $\sigma = \Psi_1 \otimes \Psi_2$ (for the principal block $\Psi_1 \otimes \Psi_2$ is free over E_{ind} so it is also flat, torsion free...).

We implement this strategy in our general context. Let $z \in Z(M)$ such that $z' = z^{-1}wzw^{-1}$ is not a compact element *i.e.* $\langle z' \rangle = \{z^k \mid k \in \mathbb{Z}\}$ is not compact. Then we have that the action of $z' = z^{-1}wzw^{-1}$ is trivial on I'_w. However, it can't' be trivial on σ as a non-zero element $f \in \sigma = \mathrm{ind}_N^M(\psi)$ which satisfies $z' \cdot f = f$ would have support containing $N\langle z' \rangle$, which is not compact modulo N. We deduce that $\mathrm{Hom}_{E_{\mathrm{ind}} \times M}(I_w, \sigma) = 0$ for all $w \neq \mathrm{id}_W$. As a consequence all morphisms in $\mathrm{Hom}_{E_{\mathrm{ind}} \times M}(\mathrm{ind}_{\bar{P}}^G(\sigma)|_M, \sigma)$ must factor through I_{Id_W} *i.e.* factor through ev_1. By Frobenius reciprocity it proves that $C_E(E_{\mathrm{ind}}) = E_{\mathrm{ind}}$. □

The latter lemma implies the existence of f' as the elements of the centre commute with E_{ind} which means $Z(E) \subseteq C_E(E_{\mathrm{ind}}) = E_{\mathrm{ind}}$. So the proof is now finished. □

Remark B.17. The proof of the last lemma is valid over any $\mathbb{Z}[1/p]$-algebra and applies for more general groups. It also brings a new method over coefficients fields such as $\mathbb{C}$ to bypass the use of generic irreducibility in order to prove:

$$\mathrm{End}_{\mathbb{C}[M/M^0] \times G}(\mathrm{ind}_P^G(\sigma\Psi)) \simeq \mathbb{C}[M/M^0]$$

where Ψ is the universal unramified character for the Levi M and $\sigma \in \mathrm{Rep}_{\mathbb{C}}(M)$ is irreducible.

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